$\mathcal{N} = 1$ and non-supersymmetric open string theories in six and four space-time dimensions

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Herrn Dipl.-Phys. Lars Görlich
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Präsident der Humboldt-Universität zu Berlin:
Prof. Dr. Jürgen Mlynek

Dekan der Mathematisch-Naturwissenschaftlichen Fakultät I:
Prof. Dr. Michael Linscheid

Gutachter:

1. Prof. Dr. Dieter Lüst
2. Prof. Dr. Stefan Theisen
3. Prof. Dr. Jan Louis

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Zusammenfassung

Die vorliegende Arbeit beinhaltet ein einführendes Kapitel über *Orbifold*-Konstruktionen in dem neben rudimentären Grundlagen bereits speziellere Themen wie *Diskrete Torsion* und *asymmetrische* Orbifold-Gruppen behandelt werden. Als Beispiele für Orbifolde werden Kompaktifizierungen auf Tori sowie das asymmetrische $T^4/\mathbb{Z}_3^{\mathrm{L}} \times \mathbb{Z}_3^{\mathrm{R}}$ Orbifold behandelt.

Danach wird eine allgemein gehaltene Einführung in *Orientifolde* gegeben, einschließlich des offenen String Sektors samt *Chan-Paton* Freiheitsgraden.

Die darauf folgenden Kapitel 4-7 behandeln von mir durchgeführte Forschungsarbeiten.

Kapitel 4 beschäftigt sich mit der Quantisierung des offenen Strings mit linearen Randbedingungen, wie sie bei Strings in elektro-magnetischen Feldern auftreten. Weiterhin wird die Quantisierung der Null- und Impuls-Moden des offenen Strings in Torus-Kompaktifizierungen durchgeführt. Außerdem wird für den Fall allgemeiner konstanter Hintergrund Neveu-Schwarz $U(1)$-Hintergrundfelder der Kommutator der Stringkoordinaten berechnet. Dieser stützt bisherige Resultate zur Nicht-Kommutativität von offenen Stringtheorien in Neveu-Schwarz Hintergründen.

Kapitel 5 gibt, zusammen mit einigen neuen Erkenntnissen, Resultate von [1] über asymmetrische Orientifolde, insbesondere deren *D*-Branen Inhalt wieder. Kapitel 6 faßt die Veröffentlichung [2] zusammen, in der untersucht wurde, inwieweit sich phänomenolgisch interessante Modelle in Orientifolden von Torus-Kompaktifizierungen finden lassen. Insbesondere tragen die D9-Branen magnetische Flüsse, womit chirale Fermionen im Spektrum auftreten. Die Rechnungen werden größtenteils im gleichwertigen, T-dualen Bild ausgeführt. In diesem ist die Anzahl der chiralen Fermionen durch die *topologische Schnittzahl* der *D*-Branen gegeben.

Existieren auf Torus-Kompaktifizierungen entweder nur nicht-chirale oder nicht-supersymmetrische Modelle, so lassen sich auf gewissen Orbifolden beide Eigenschaften miteinander vereinbaren. Kapitel 7 behandelt das $\bar{\sigma}\Omega$-Orientifold auf einem $T^6/\mathbb{Z}_4$ Orbifold. Als besonders interessantes Beispiel wird ein supersymmetrisches $U(4) \times U(2)_{\mathrm{L}}^3 \times U(2)_{\mathrm{R}}^3$ Modell vorgestellt, daß durch Einschalten geeigneter Hintergrundfelder in der effektiven Niederenergie-Wirkung auf ein Modell gebrochen wird, daß dem *MSSM* (minimalem supersymmetrischen Standard Modell) sehr ähnlich ist. Dieses Kapitel basiert auf unserer Publikation [3].

Ferner ist der Arbeit ein Anhang beigefügt, der einige der verwendeten Formeln sowie Beweise zu zwei Sätzen enthält, die im Text verwendet wurden.

Schlagwörter:
String Theorie, Orientifolds, offene Strings, D-Branen, supersymmetrische Modelle, String Phänomenologie

Abstract

This thesis contains an introductory chapter on orbifolds. Besides rudimentary basics we discuss more advanced topics like *discrete torsion* and *asymmetric* orbifold groups. As examples we investigate torus compactifications and an asymmetric $T^4/\mathbb{Z}_3^{\mathrm{L}} \times \mathbb{Z}_3^{\mathrm{R}}$ orbifold.

The following chapter explains the foundations of orientifolds, including open strings with *Chan-Paton* degrees of freedom.

Chapters 4-7 present own research.

In chapter 4 we quantize open strings with linear boundary conditions, as they show up in electro-magnetic fields. We quantize the zero- and momentum-modes for toroidal compactifications, too. As an application we calculate the commutator of the coordinate fields in the case of general constant Neveu-Schwarz $U(1)$-field strengths. Thereby we confirm previous results on *non-commutativity* of open string theories in Neveu-Schwarz backgrounds.

Chapter 5 reviews the results of a former publication [1] on asymmetric orientifolds, supplemented by some recent insights in connection with the preceeding chapter.

Chapter 6 is a summary of [2]. In this publication we investigated to what extend one can build phenomenologically interesting models from toroidal orientifolds. By turning on magnetic fluxes on D9-branes we induce chiral fermions. Most calculations are performed in an (equivalent) T-dual picture. Here the number of chiral fermions is given by the topological intersection number of D-branes.

In orientifolds of toroidal compactifications one obtains either non-chiral or non-supersymmetric orientifold solutions. However both properties can be reconciled in orientifolds that are obtained from specific supersymmetric orbifold compactifications. In chapter 7 we present the $\bar{\sigma}\Omega$-Orientifold on a $T^6/\mathbb{Z}_4$ orbifold. As a very attractive example we investigate a supersymmetric $U(4) \times U(2)^3_{\mathrm{L}} \times U(2)^3_{\mathrm{R}}$ model that is broken to an *MSSM* [1] -like model by switching on suitable background fields in the *low energy effective action*. This chapter is based on our publication [3].

The thesis is supplemented by an appendix with formulæ applied in the text, as well as proofs to two theorems that were used as well.

Keywords:
string theory, orientifolds, open strings, D-branes,supersymmetric models,string phenomenology

[1] “MSSM”= minimal supersymmetric Standard Model

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Chapter 1

Introduction

In this chapter we will give a short motivation for string theory and its supersymmetric extension. By doing so we will expose some of the basic ideas underlying this theory. As the main part of the thesis and the whole part of our own research presented here deals with open strings, we devote a section to this topic as well. In chapter 6 and 7 we will investigate the chance to find realistic models in a special class of unoriented open-string theories. Therefore we will make some comments on these orientifolds as well. Since we have included two more extended chapters on orbifolds and orientifolds in this thesis, this introductory chapter is rather condensed, only concentrating on the rudiments without going into details. General texts on string theory are the classical work of Green, Schwarz and Witten [4, 5], the book of Lüst and Theisen [6] and the two volumes by Polchinski [7, 8]. The latter reference is especially interesting as it includes a chapter on D-branes and gives a D-brane interpretation of orientifolds. The book of Bailin and Love provides a good introduction to both supersymmetric field theory and superstring theory (cf. [9]).

1.1 String theory

String theory is a quantum theory of string-like (i.e. one-dimensional) objects. Even though it has many similarities to quantum mechanics of point-like particles and known quantum field-theories, there are many striking differences as well. We will recall some principles used to quantize classical systems and try to apply those to the string as well. It turns out that string theory is in some respect much more restrictive, but offers a lot of promising features at the same time. The most interesting one is surely that string theory is automatically a (presumably) consistent theory of quantum gravity, already at the perturbative level. Other interesting features are gauge-symmetries in the low-energy effective action of a huge class of string theories. A third feature is that chiral fermions appear in many string theories, thereby making string theory a good candidate for a unified theory of nature.

1.2 Quantization of classical strings

In many approaches to quantum theory one starts with a classical system in canonical formalism. A main ingredient in this formulation is the symplectic phase space of the system, which is the cotangent bundle $T^*(\mathcal{M})$ of a manifold $\mathcal{M}$. For example in classical mechanics $\mathcal{M}$ is the $3n$-dimensional manifold describing the positions of n point-like particles. In a system with infinitely many degrees of freedom (dofs) one often does not bother about the precise structure of $\mathcal{M}$, which is in this case infinite, too. On the (finite-dimensional) manifold $T^*(\mathcal{M})$ an algebra of C^∞-functions exists, which we denote by $\mathcal{T}\big(T^*(\mathcal{M})\big)$. What is now important, is that the symplectic structure of the phase-space induces a bilinear map from the space of C^∞-function to itself. This map is commonly known as the *Poisson-bracket*:

$$\begin{aligned} \mathcal{T}\big(T^*(\mathcal{M})\big) \times \mathcal{T}\big(T^*(\mathcal{M})\big) &\to \mathcal{T}\big(T^*(\mathcal{M})\big) \\ (f,g) &\mapsto \{f,g\}_{\mathrm{PB}} \end{aligned} \tag{1.1}$$

By quantizing the system the classical algebra of observables $\mathcal{T}\big(T^*(\mathcal{M})\big)$ gets exchanged by some operator algebra.[1] Since both the Poisson-bracket and the *commutator* of an operator algebra share three important properties (bilinearity, anti-symmetry and Jacobian identity) it is natural to map the Poisson-bracket of two functions f, g to the commutator of the corresponding operators $\hat{f}, \hat{g}$ in the operator algebra.[2]

In contrast to point-particles, strings are one-dimensional objects, but they admit a classical description in terms of a *Lagrangian* (density), a derived symplectic form and a Hamiltonian as well. Functions on its phase space, especially the coordinate functions of the string and the canonical momentum might be substituted by operators as well, thereby preparing the grounds for a quantum theory of strings. Quantum mechanics has still a richer structure. The probability interpretation of quantum mechanics requires that the operator algebra has to act on some vector space which admits a *hermitian* scalar product. In the best case the vector space is closed (w.r.t. the scalar product) i.e. a *Hilbert space*. Observable quantities are the spectra (Eigenvalues) of operators corresponding to classical quantities, and as these are real, one requires these operators to be self-adjoint.

While in the simplest case of point particles such a Hilbert space and operator algebra are relatively easy to find,[3] it turns out that more complicated systems (i.e. infinitely many degrees of freedom, like quantum field-theories) a direct map from the classical- to the quantum-system is often problematic. This may have many reasons and up to now there exists no prescription, how to quantize an arbitrary classical system. For example in classical field theory

[1]These statements should be taken with a grain of salt. We are not very precise about the operator algebra involved, and especially not about the map form $\mathcal{T}\big(T^*(\mathcal{M})\big)$ to this algebra. Furthermore there might appear additional subtleties.

[2]Usually one maps the Poisson bracket to $\hbar/i$ times the commutator, but the normalization is somehow redundant.

[3]Even though one encounters already ambiguities in the map from the function- to the operator-algebra (e.g. ordering of operators).

there is no obstruction in multiplying two fields $\phi(x)$, $\psi(y)$, even if both fields have the same argument $x = y$. In quantum field theory the product of the corresponding fields (which are in a naive approach operator valued functions) with coinciding arguments $x = y$ will in general be singular, i.e. not properly defined. In many quantum field theories, these infinities can be "regularized" and a program called renormalization expresses the parameters that are introduced in the regularization procedure by physical, i.e. measurable quantities.

While the problem discussed above usually becomes important if one considers some kind of interaction, it might already be a challenge to find the correct Hilbert space even if one neglects interactions. This is the case for the electromagnetic Maxwell field, but also for the string. It is well known that a naive quantization of the electromagnetic field A^μ induces states of negative norm due to the minkowskian scalar product in space-time. Some classical equalities can lead to contradictions if directly translated into operator equations. This problem can be solved by the so called *Gupta-Bleuler* quantization. The idea is to split the state-space into a physical one, and a redundant space. The physical Hilbert-space is obtained by requiring that the classical conditions are fulfilled by the positive frequency (or annihilation-) part of the corresponding quantum fields:[4]

$$F_{\text{class}} = 0 \qquad \Rightarrow F^{(+)}_{\text{qm}} |\psi, \text{phys}\rangle = 0 \tag{1.2}$$

For Gupta-Bleuler quantization of the electromagnetic field, F equals the (four-dimensional) divergence of the vector potential: $F = \partial_\mu A^\mu = 0$. These conditions are linear, therefore the physical space is a linear subspace of the vector space from which one starts. This physical subspace has a positive *semi*-definite norm. In electrodynamics there is still a redundancy in this subspace. Physical states belong to equivalence classes of the subspace and measurable quantities are not affected by the representative chosen. The redundancy corresponds to the (unphysical) longitudinal and time-like polarization part whose *non-zero* excitations are of zero-norm. Requiring that the longitudinal part admits a zero-excitation contribution with non-vanishing norm ensures that the state is normalizable, thereby turning the space of equivalence classes into a (pre-) Hilbert space, i.e. a vector space of positive definite scalar product. It is very assuring, that the (non-physical) longitudinal excitations decouple from the S-matrix. The gauge invariance of electromagnetism might be regarded as the origin of the split. Analogous features are encountered if one quantizes more general quantum field theories (QFTs) like non-abelian gauge theories and even more interesting for us: Something similar occurs for the string as well.

The classical action for the string is proportional to the area of the string *world sheet*. The string world sheet is the two-dimensional analog of the world line for point-particles. The name *Nambu-Goto action* is devoted to its inventors:

$$S_{\text{NG}} = -\frac{1}{2\pi\alpha'} \int d^2\sigma \sqrt{-\det_{\alpha\beta}\Big(\big(\partial_{\sigma^\alpha} X^\mu\big)\big(\partial_{\sigma^\beta} X_\mu\big)\Big)} \tag{1.3}$$

[4] As the creation-part $F^{(-)}$ is the hermitian conjugate of $F^{(+)}$ matrix elements between physical states involving *normal-ordered* combinations of $F^{(\pm)}$ will vanish.

α' is the so called *Regge-slope*.[5] The solutions of the equations of motion (eoms) for the embedding fields X^μ justify this identification. The eoms of the Nambu-Goto action are highly nonlinear, even if the background on which the string propagates is flat and consequently difficult to solve. Therefore the following more tractable action (*Polyakov-action*) was proposed:[6]

$$S_{\mathrm{P}} = -\frac{1}{4\pi\alpha'} \int d^2\sigma \sqrt{-\det h}\, h^{\alpha\beta} \left(\partial_{\sigma^\alpha} X^\mu\right)\left(\partial_{\sigma^\beta} X_\mu\right) \tag{1.4}$$

$h^{\alpha\beta}$ is a metric defined on the world sheet. Solving the eoms for $h^{\alpha\beta}$ and reinserting the (formal) solution into the Polyakov action (1.4) results in the Nambu-Goto action (1.3). At this classical level the two actions are therefore equivalent. It is however an open issue to show that they coincide if one takes the quantum fluctuations in $h^{\alpha\beta}$ into account as well. Taking the Polyakov-action as our starting point, there are several possibilities to quantize the theory, all (of the explicitly known) leading to the same result. We restrict to the case of flat space-time metric, which implies the maximal (i.e. D-dimensional) Poincaré-invariance of the string Lagrangian-density and action. Like the Nambu-Goto action, S_{P} is invariant under diffeomorphisms of the two-dimensional world-sheet. In addition, the Polyakov-action is invariant under a Weyl rescaling of the world-sheet metric:[7]

$$\begin{gathered} X'(\tau,\sigma) = X(\tau,\sigma) \\ h_{\alpha\beta} = e^{2\omega(\tau,\sigma)} h_{\alpha\beta}, \quad \text{with } \omega(\tau,\sigma) \text{ arbitrary} \end{gathered} \tag{1.5}$$

Reparameterization invariance is sufficient to transform the metric h (at least locally) to diagonal form proportional to $\mathrm{diag}(-1,1)$. Using in addition Weyl invariance allows one to obtain the following gauge:

$$h_{\alpha\beta} = \eta_{\alpha\beta} \qquad \eta_{\alpha\beta} \equiv \mathrm{diag}(-1,1) \tag{1.6}$$

The gauge (1.6) is called *conformal gauge* because it is preserved by a combination of general conformal transformations (which leave $h_{\alpha\beta}$ invariant up to a scale factor) and a subsequent Weyl transformation, that rescales the metric to its original form (1.6). The quantization procedure might be performed as follows: First one solves the eoms for the X^μ coordinate fields which become wave equations for flat space-time metric:

$$\partial^2_{\sigma^0} X^\mu = \partial^2_{\sigma^1} X^\mu \quad \mu = 0 \ldots D \tag{1.7}$$

Furthermore the X-fields are subjected to boundary conditions. The most common boundary conditions are periodic ones in the world sheet coordinate $\sigma \equiv \sigma^1$ ($\tau \equiv \sigma^0$):

$$X^\mu(\tau, \sigma + 2\pi) = X^\mu(\tau,\sigma) \tag{1.8}$$

[5] In general the Regge-slope is defined as the maximal angular momentum per energy^2.

[6] This action was found by Brink, Di Vecchia, Howe, Deser and Zumino. Polyakov used it to perform path-integral quantization.

[7] The Weyl-invariance would not be present for higher or lower dimensional objects like membranes or point-particles.

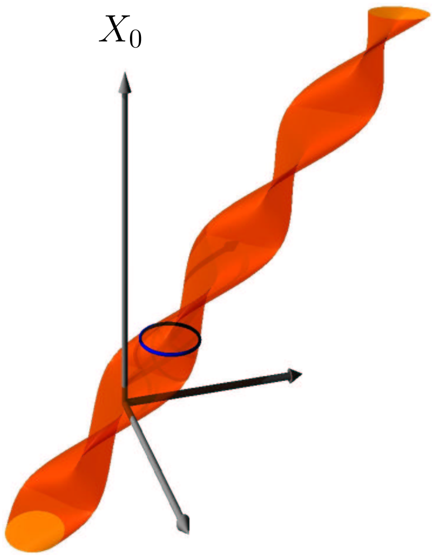

Figure 1.1: Closed-string (blue) evolving in time. The world-sheet, which is a classical solution is indicated in transparent orange.[8]

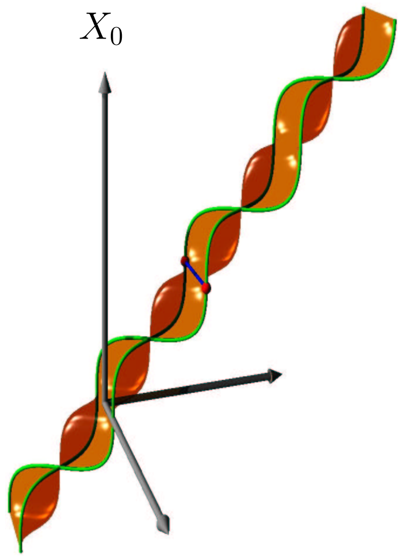

Figure 1.2: Open-string (blue) evolving in time. Both world-sheet boundaries (green) belong to the same stack of D-branes. This classical solution can be associated with a gauge-boson of the quantized theory.[8]

and open string boundary conditions which are of Neumann-type:

$$\partial_\sigma X^\mu(\tau,\sigma)\big|_{\sigma=0,\pi} = 0 \tag{1.9}$$

Classical closed and open strings fulfilling these eoms and boundary conditions are depicted in figure 1.1 and 1.2.[8] Then one quantizes the classical degrees of freedom. The world-sheet Hamiltonian suggests a splitting into creation and annihilation operators. Like in the case of electrodynamics one encounters however necessarily states of zero-, and even worse: negative-norm. Much alike in the case of electrodynamics one now tries to impose further classical constraints. The additional classical conditions stem from the eoms of the world-sheet metric $h_{\alpha\beta}$ i.e. the vanishing of the world sheet energy-momentum tensor T. Its trace vanishes already by the Weyl-invariance of the classical action.[9] T can be expressed in terms of X and consequently in terms of operators. The Fourier components are the famous Virasoro generators L_n. In calculating the Pois-

[8]However the depicted world-sheet does not fulfill the classical constraint equations. These would imply that no oscillator modes are excited for a light-like center of mass momentum p. The normal-ordering of the quantum theory enforces however that oscillators are excited for $p^2 = 0$.

[9]However the Weyl-symmetry might get spoiled by quantum effects. Actually in order that Weyl-anomalies are absent in the path integral formalism, one is restricted to $D = 26$ space-time dimensions under certain assumptions on the background. This is especially true for constant background fields.

son brackets of the classical Virasoro generators[10] and comparing it with the quantum mechanical commutator obtained by expressing the L_n in terms of operators encountered in quantizing X, one discovers a c-number anomaly, the so called *Virasoro anomaly*. Unlike to electrodynamics, the condition that the positive frequency part of the energy-momentum tensor $T^{(+)}$ has to vanish on the physical Hilbert-space $\mathcal{H}_{\text{phys}}$, does not remove negative-norm states from $\mathcal{H}_{\text{phys}}$. It does so if a previously obtained normal ordering constant a equals one and if the space-time dimensions D equals 26. Higher space-time dimensions are not possible, while there are examples for $a < 1$ and $D < 26$ that emerge from projecting the 26-dimensional theory to lower dimensions. Looking at the massless spectrum, one can however not single out the case $D = 26$ and $a = 1$. This might be done by looking at vertex operators, or probably more conveniently: to choose the method of *light-cone quantization*.

In light-cone quantization one transforms the time-coordinate X_0 and one arbitrary space-coordinate, say X_1 to new coordinates $X_\pm = 1/\sqrt{2}(X_0 \pm X_1)$. The constraints $T = 0$ take the following simple form in light-cone coordinates:

$$\begin{aligned}
\partial_\tau X_+ \partial_\tau X_- + (\tau \leftrightarrow \sigma) &= \frac{1}{2} \sum_{i=2}^{D} (\partial_\tau X_i)^2 + \tau \leftrightarrow \sigma \\
\partial_\tau X_+ \partial_\sigma X_- + (\tau \leftrightarrow \sigma) &= \frac{1}{2} \sum_{i=2}^{D} (\partial_\tau X_i)(\partial_\sigma X_i)
\end{aligned} \tag{1.10}$$

The interesting observation is that X_- appears only linear in the above constraint equations. If we would be able to bring X_+ to a particular simple form, i.e. one which is linear in τ (or σ), we could solve these equations directly. Due to the formerly mentioned residual conformal symmetry (which leaves the form of the gauge fixed action and metric h invariant) and due to the fact that the X-fields have the same periodicity as the conformal transformations, which are harmonic functions on the world-sheet as well, this is indeed possible. The resulting spectrum can be shown to be ghost-free. However Lorentz-symmetry is no longer manifest. It turns out that in general the Lorentz-symmetry is plagued with anomalies, except for the case of space-time dimension $D = 26$.

Another method to quantize strings is the *path-integral* approach. It is relatively complicated, although leading to most insights in mathematical respects. In path-integral formalism the absence of quantum anomaly in Weyl-transformations (1.5) restricts the space time dimension to $D = 26$ which also removes a possible BRST-anomaly.[11]

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[10] The Virasoro generators together with its commutators are called the (central extension of the) *Virasoro algebra*.

[11] Anomalies in symmetries that are used to split the Hilbert space into a physical and an unphysical part (and this is exactly what the BRST-symmetry is used for) would indicate that this split is ruined by quantum corrections.

1.3 String theory as a theory of quantum gravity

Up to now we explained how string theory is quantized in principle, and how the corresponding Hilbert space can be obtained. We saw that this Hilbert space only exists for bosonic strings moving in $D = 26$ space-time dimensions, which already puts surprisingly many constraints on the geometry. (For supersymmetric strings the number of flat dimensions turns out to be 10.) Up to now we restricted to a flat target space. However the form of the string action suggests some generalization. If one computes the spectrum, one sees that it is quantized due to the constraint on the energy-momentum tensor, or its Fourier-components, the L_n's. The linear mode $p_{\rm com}$ of the fields

$$X(\tau, \sigma) \sim x_{\rm com} + \tau \cdot p_{\rm com} + \text{oscillator modes} \tag{1.11}$$

is interpreted as the center of mass (com.) momentum. Its (minkowskian) square determines the mass of the state. It turns out that for the bosonic string on flat space-time there exist tachyons for both open and closed strings. The mass^2= 0 level consist of an excitation that has exactly the degrees of freedom of a $U(1)$-gauge field for the open string. The closed string mass^2= 0 can be identified with a scalar (the dilaton), an antisymmetric tensor, and a traceless symmetric tensor, the latter interpreted as the graviton. This makes string theory particularly interesting. String theory gives further evidence that this identification is justified. According to the massless particle content, it is suggestive to include further terms in the Polyakov action, which are compatible with two-dimensional diffeomorphism and Weyl invariance at the classical level:[12]

$$S_\sigma = -\frac{1}{4\pi\alpha'} \int d^2\sigma \sqrt{-|h|} \Big(\big(h^{\alpha\beta} G(X) + \epsilon^{\alpha\beta} B(X)\big)_{\mu\nu} \frac{X^\mu}{\partial\sigma^\alpha} \frac{\partial X^\nu}{\partial \sigma^\beta} + \alpha' R(h) \Phi(X) \Big) \tag{1.12}$$

G is the space-time dependent D-dimensional metric, B the antisymmetric tensor, Φ the dilaton field, while R is the two dimensional Ricci-scalar. The background fields in the above action might be interpreted as coherent states of strings, which might be represented by insertions of vertex operators into the path-integral.[13] The action (1.12) describes a coupled two-dimensional field theory with the couplings G, B and Φ depending on the fields X^μ in a possibly non-linear way. (Such an action is therefore called a non-linear σ-model.) These coupling functionals will admit β functions like any coupling in a QFT. Weyl invariance at the quantum level requires, that these β-functions vanish. It is possible to obtain the β functions (of the two dimensional world-sheet theory) corresponding to the three fields G, B and Φ as eoms of the following D-

[12]We neglect for the moment a possible boundary action that would include a vector-potential A^μ corresponding to the open string massless mode.

[13]States can be created by so called *vertex operators*. This is similar to the case of QFT, where *in* and *out* states are created by corresponding fields. Vertex operators play an essential role in calculating string interactions.

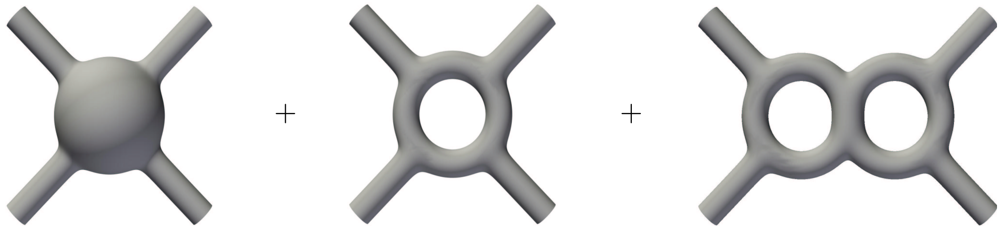

Figure 1.3: First three terms of the string perturbation series with four external closed string states involved

dimensional action:[14]

$$S_{\mathrm{S}} = \frac{1}{2\kappa_0^2}\int d^Dx\sqrt{-G}e^{-2\Phi}\Big(-\tfrac{2(D-26)}{3\alpha'}+R(G)-\frac{1}{12}H\wedge{*H}+4d\Phi\wedge{*d\Phi}+O(\alpha')\Big) \tag{1.13}$$

H is the field strength of the antisymmetric tensor: $H = dB$. Upon a Weyl rescaling of the metric $\widetilde{G}(x) = \exp(2\omega(x))G(x)$, $\omega(x) = 2(\Phi_0 - \Phi(x))/(D-2)$ together with the induced transformation of the Ricci scalar $R(G)$ and a further field redefinition of the dilaton $\widetilde{\Phi} = \Phi(x) - \Phi_0$ the action (1.13) becomes ($\kappa = \kappa_0 \exp(\Phi_0)$):

$$\begin{aligned} S_{\mathrm{E}} = \frac{1}{2\kappa^2}\int d^Dx\sqrt{-\widetilde{G}}\Big(&-\tfrac{2(D-26)}{3\alpha'}e^{\frac{4\widetilde{\Phi}}{D-2}} + R(\widetilde{G}) \\ &- \tfrac{1}{12}e^{-\frac{8\widetilde{\Phi}}{D-2}}H\wedge\tilde{*}H - \tfrac{4}{D-2}d\Phi\wedge\tilde{*}d\Phi + O(\alpha')\Big) \end{aligned} \tag{1.14}$$

Because the action (1.14) is the Einstein-Hilbert action of gravity supplemented with some additional fields, the metric $\widetilde{G}$ is denoted as the *Einstein metric*, while G is called *string metric*. The action (1.14) governing the background fields is the most impressive justification for identifying the symmetric traceless mode of the perturbative closed string with the quantum excitation of the graviton field.

String theory perturbation series are defined as integrals over the moduli space of Riemann surfaces with insertions of vertex operators (whose positions are also moduli).[15] The vertex operators correspond to external (i.e. incoming and outcoming) particles. For closed strings there exists only one diagram at a given genus. It includes implicitly all possible string excitations in the internal part of the diagram. A four-particle closed string scattering process is depicted up to third order in figure 1.3. Besides the sphere it includes a torus with one and another torus with two handles. From the aspect of simplicity (i.e. one diagram at each level of closed-string perturbation series, and still comparatively few, if one includes open strings and unoriented diagrams) string

[14]The β-function leading to this action were obtained by expanding the background field up to first order in coordinate fields X. Higher order corrections are included in $O(\alpha')$.

[15]To be more precise, one only integrates over a region in moduli space, which is not connected to another one by an holomorphic transormation.

theory is very economic. If one considers for example all diagrams contributing to the one-loop level of electron-electron scattering ($e^-e^- \to e^-e^-$) one gets a variety of diagrams which are shown in figure 1.4. We even suppressed the

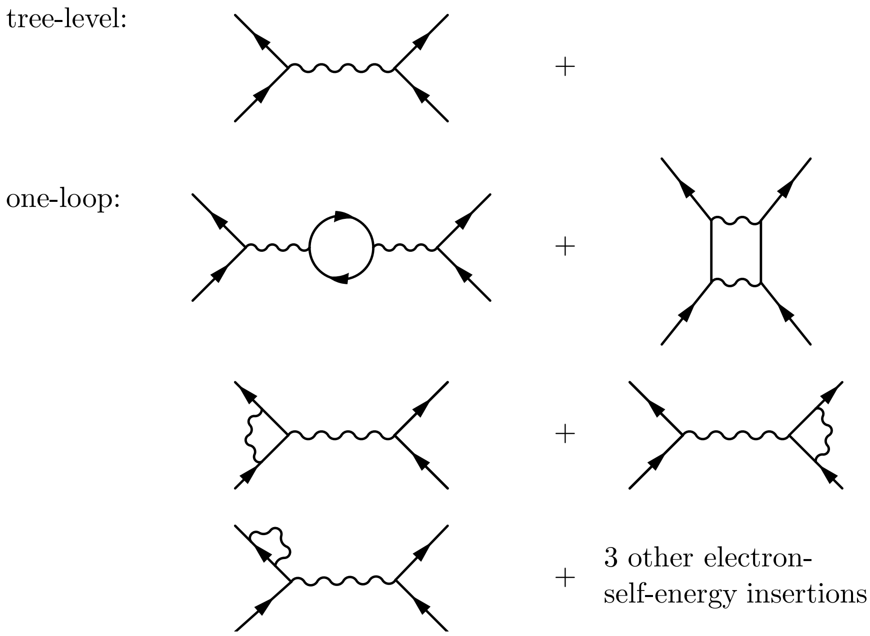

Figure 1.4: Perturbative expansion of electron-electron scattering in QED with one fermion generation.

different combinations of legs, which would lead to a multiple of the depicted diagrams. Similar combinatorics occur however in string theory as well, if several open strings participate as external states. We also have to admit that the actual calculation of the few string-diagrams is a highly non-trivial task, at least at higher loop orders, or for many external strings participating.

As string theory has the graviton in its spectrum, we can (at least formally) calculate scattering amplitudes that include this excitation as an external state. In loop diagrams the graviton is implicitly included as an internal state as well. If one tries to include the graviton in conventional QFT, one is lead to serious problems in performing the perturbative expansion (while little is known about a non-perturbative treatment of a QFT of gravitation). This is due to the fact that the usual renormalization program, that allows to absorb all divergencies in a *finite* number of (measurable) constants, fails. This kind of non-renormalizibility can be traced back to the fact that the gravitational constant has negative mass dimension (in units where $\hbar = 1$).

In string theory the problems with (UV-) divergencies are circumvented in a very elegant way: In field theory the dangerous divergencies are UV-divergencies, i.e. divergencies that appear for high momenta. In coordinate space an one-loop UV divergence would correspond to the limit, where the loop size shrinks to zero. In principle this divergence is also seen in string theory, if one considers the limit, when the modular parameter (or complex structure) that describes the shape of the torus, approaches zero. However the symmetries of string theory, in this case: modular invariance, require that one only integrates the modular parameter over a region that describes inequivalent tori. A convenient integration region for the torus modulus is given by the shaded region in figure 2.2 on page 28. The regions including possible singularities are explicitly excluded by this choice. Therefore one-loop torus amplitudes are UV-finite (Shapiro [10]). Modular invariance extends to higher loop-levels as well. Even though not strictly proven yet, it is believed that the finiteness of string-scattering amplitudes extends to all orders of string perturbation theory. It would imply that string theory describes perturbative quantum gravitation. This is one (maybe even the strongest) motivation to consider string theory as a unifying theory.

Up to now we concentrated on the massless modes of string theory. There is still an infinite tower of massive states. For flat backgrounds the different mass-levels are equally spaced. Bosonic string theory contains a tachyon, which seems to indicate an instability. Some researchers undertake however considerably effort in order to stabilize the theory via some kind of tachyon condensation. Another way out of this problem is to look for a string theory where tachyons are manifestly absent. This is the case for:

1.4 Supersymmetric string theories

An obvious shortcoming of the bosonic string is the absence of space-time fermions in its spectrum.[16] Furthermore it is desirable to have a supersymmetric theory in space time, at least to a good approximation. In any phenomenologically relevant theory this supersymmetry has to be broken at some scale, and if string-theory serves as a unifying theory, this breaking must be compatible with the underlying principles of string theory. In this thesis, we will not concentrate on the supersymmetry breaking mechanism.

To include space-time fermions, the bosonic string action 1.4 is extended by terms that involve fermionic dofs. There are two common ways to achieve this. One is the *Green-Schwarz* (GS) superstring(-formalism) [14, 15]. Instead of the purely bosonic action one considers (α' set to 1/2):

$$S_1 = -\frac{1}{2\pi}\int d^2\sigma\sqrt{-\det h}\, h^{\alpha\beta}\,\Pi^\mu_\alpha(\Pi_\beta)_\mu, \qquad \Pi^\mu_\alpha = \partial_\alpha X^\mu - i\bar\theta^A\Gamma^\mu\partial_\alpha\theta^A \quad (1.15)$$

In this approach $\theta^A, A = 1\ldots N$ are N space-time spinors. Each of the spinor components is a world sheet scalar. This action is reparameterization invariant. Requiring a so called κ-symmetry in order to reduce the number of fermionic dofs lets one introduce an additional action piece S_2. κ-symmetry restricts then the maximal number of spacetime spinors to $N = 2$. Requiring S_2 to be supersymmetric reduces the possible space-time dimensions considerably. The quantized version singles out $D = 10$ in which case already supersymmetry of the action S_2 requires that both spinors θ^1 and θ^2 are of *Majorana-Weyl* type.

[16] It has however been suggested that the bosonic string includes even the supersymmetric string-theories in a rather subtle manner (cf. [11, 12, 13]).

The Green-Schwarz formalism has the advantage to be manifestly supersymmetric in space-time. However the resulting eoms are extremely complicated, as they are non-linear. They can be drastically simplified by choosing light-cone gauge, and simplifying the X_+-coordinate as in the bosonic case. As the Lorentz-algebra can only be realized in $D = 10$ space-time dimensions this case is considered as the only consistent. Depending on the relative handedness of θ^1 and θ^2 one obtains either the Type IIA (θ^1 and θ^2 have opposite chirality) or Type IIB (θ^1 and θ^2 have equal chirality). In the case of Type I both spinors are identified (by moding out the world sheet-parity Ω). We come back to this case in section 1.6. There is still a third kind of ten-dimensional superstring known, the *heterotic string*. As its name suggests, the construction of the heterotic string is composed from several pieces. The heterotic string takes advantage from the fact that the closed string-states can be decomposed into left and right moving parts. The same is true for the fields. Roughly speaking, the theories considered so far are constructed from a tensor product of left- and right moving degrees of freedom. This does not mean that the resulting theories are tensor products as well, since in general some additional conditions have to be imposed. The probably most famous construction of the heterotic string starts from ten-dimensional superstring of one (say the right-moving) sector and the 26-dimensional bosonic string in the other (here: left-moving) sector. In order to have a sensible space-time interpretation, one compactifies the sixteen surplus bosonic dimensions. Especially one considers flat toroidal compactifications that are obtained by identifying points $x \sim x + 2\pi\gamma$ with γ a vector of a sixteen dimensional lattice Λ^{16}. Associated with the torus lattice Λ is an even self dual lattice, the so called *Narain-lattice* Γ^{16}, which is in general not unique (cf. chapter 2). However there are only two 16-dimensional Narain-lattices:

1. Γ^{16} is the weight-lattice of the $Spin(32)/\mathbb{Z}_2$ [17]
   $\big(\Gamma_{\text{root}}(SO(32)) \subset \Gamma_{\text{weight}}(Spin(32)/\mathbb{Z}_2)\big)$

2. $\Gamma^{16} = \Gamma^8 \times \Gamma^8$ with Γ^8 the root-lattice of the E_8 Lie-algebra

What makes heterotic string-theories so extremely interesting is the fact that they admit non-abelian Lie-algebras as gauge-symmetries of their low-energy-effective field theory. In the first case this symmetry is $SO(32)$ while in the second it is $E_8 \times E_8$. These symmetries are also manifest in the operator product expansion (OPE) of the formerly mentioned vertex operators.[18] Compactifications of the heterotic string to four space-time dimensions have led to many interesting models, especially such that come pretty close the Standard Model (SM) of electro-weak and strong interactions.

[17] By this we mean the sub-lattice of the weight-lattice of $Spin(32)$ which is generated by the weights of *one* spinor representation together with the the roots of $SO(32)$. (Remember that the weight lattice of $Spin(32)$ consist of four conjugacy classes: adjoint (i.e. roots), vector, spinor, spinor'.)

[18] In the heterotic string there exist currents on the world-sheet (associated with charges) that build up the corresponding *Kac-Moody algebras.* A Kac-Moody algebra is an infinite dimensional extension of a Lie algebra.

In parallel to the Green-Schwarz superstring there exists the so called *Neveu-Schwarz-Ramond* (NSR) superstring.[19] It turns out that the GS- and the NSR-superstring describe the same physics, though they use other formalisms. While the GS-superstring exhibits manifest space-time supersymmetry, its covariant quantization is not at all obvious. The NSR superstring however can be quantized by path-integral formalism in parallel with the bosonic string, up to some generalizations. It becomes space-time supersymmetric, if one imposes the so called *Gliozzi-Scherk-Olive* (GSO) projection (which is absent in GS-formalism) [16, 17]. From the world-sheet point of view the bosonic string action 1.4 might be considered as a two dimensional gravity theory (after inclusion of the $R(h)$-term like in (1.12)) coupled to D world-sheet scalars X^μ. It is now quite natural (and actually necessary in order to use path-integral formalism in a subsequent analysis) to extend this theory to $N = 1$ local supersymmetry (or: $N = 1$ supergravity) on the world sheet. Supersymmetrizing the scalar part of the bosonic action is achieved by adding the following term to the Polyakov action (1.4):

$$S_F = -\frac{1}{2\pi\alpha'}\int d^2\sigma\sqrt{-\det h}\left(i\bar{\psi}^\mu\rho^\alpha\partial_{\sigma^\alpha}\psi_\mu + F^\mu F_\mu\right) \tag{1.16}$$

Some comments are in order: Each ψ_μ, $\mu = 0, \ldots, D-1$ is a two dimensional (world-sheet) Majorana-spinor. The D world-sheet spinors ψ_μ make up a space-time Lorentz-vector. (This is in contrast to the GS-formalism, where the θ^A are space-time spinors as well as world-sheet scalars.) The F^μ are auxiliary fields which are needed to realize the off-shell supersymmetry-algebra. Their eoms however require them to vanish on-shell. Each of the F^μ is a world-sheet scalar while in total they make up a D dimensional space-time Lorentz-vector. The metric h can be expressed by world-sheet Vielbeins $e^\alpha{}_a$ (or more precisely as they live in *two* dimensions: by *Zwei*beins.):

$$e^\alpha{}_a e^\beta{}_b h_{\alpha\beta} = \eta_{ab} \qquad a, b, \alpha, \beta \in \{0, 1\}, \qquad \eta = \mathrm{diag}(-1, 1) \tag{1.17}$$

As $GL(d, \mathbb{R})$ does not admit finite-dimensional spinor-representations, Vielbeins are a way to define spinors on curved space-time, in our case: on a curved world-sheet. The two dimensional matrices ρ^α are obtained from the two dimensional Dirac matrices (cf. eq. (4.5),(4.6), p. 88) by:

$$\rho^\alpha \equiv e^\alpha{}_a \rho^a \tag{1.18}$$

The sum of the bosonic action (1.4) and the fermionic action (1.16) does not yet admit *local* supersymmetry. This goal is achieved by adding a third piece to the action:

$$S_3 = \frac{i}{4\pi\alpha'}\int d^2\sigma\sqrt{-\det h}\,\bar{\chi}_\alpha\rho^\beta\rho^\alpha\psi^\mu\left(\partial_{\sigma^\beta}X_\mu - \tfrac{i}{4}\bar{\chi}_\beta\psi_\mu\right) \tag{1.19}$$

χ_α is the superpartner of the world-sheet metric $h_{\alpha\beta}$ (or of the Zweibein $e^\alpha{}_a$). It has a world-sheet vector- and a world-sheet spinor-index. The resulting action has a variety of symmetries:

- Local world-sheet supersymmetry

[19] The NSR formalism was developed before the GS-formalism

- Local Weyl-invariance (The Weyl transformation rescales also the Majorana-fermions ψ^μ and the gravitino χ_α besides the Zweibein $e^\alpha{}_a$)
- Local super-Weyl-invariance ($\lambda(\tau,\sigma)$ a Majorana spinor parameter):

$$\delta_\lambda \chi_\alpha = \rho_\alpha \lambda \qquad \delta_\lambda(\text{others}) = 0 \tag{1.20}$$

- World-sheet (or: two-dimensional) Lorentz-invariance
- World-sheet reparameterization- (or: diffeomorphism-) invariance

Very similar to the purely bosonic case, one could use some of the symmetries to eliminate some degrees of freedom. Using local supersymmetry, reparameterization and Lorentz-invariance, one can reduce the two-dimensional supergravity action to a much simpler action (cf. eq. (4.1), (4.3), p. 88). The corresponding gauge where the gravitino is efficiently eliminated while h is brought to the standard minkowskian form is called *superconformal-gauge*. Besides the conformal symmetry encountered in the bosonic conformal gauge, this action admits a further symmetry, generated by the fermionic current T_F. T_F is determined by varying the (non-gauge fixed) action with respect to the gravitino $\bar\chi^\alpha$:

$$T_F = \frac{2\pi}{i \det e} \cdot \frac{\delta S}{\delta \bar\chi} \tag{1.21}$$

Along the lines of light-cone gauge in the bosonic case, one can eliminate in addition the ψ^+ component from the world-sheet Majorana spinor.[20] In contrast to the bosonic case the critical space-time dimension D turns out to be *ten*, rather than 26. (The formerly mentioned normal ordering constant a equals now *one half* in the bosonic (Neveu-Schwarz) sector instead of one, while it is *zero* in the fermionic (Ramond) sector. These sectors will be explained below.) Equivalent results can also be derived via path-integral quantization.

There is still a peculiar feature in the NSR superstring which we now want to address: So far we have not specified the boundary conditions of the Majorana spinor ψ. For several reasons, and the most striking one is *modular invariance* (to be explained in chapter 2), one is forced to allow ψ to be both periodic and antiperiodic for closed strings:[21]

$$\psi^\mu_\pm(\tau,\sigma) = \kappa_\pm \psi^\mu_\pm(\tau,\sigma+2\pi) \qquad \kappa_\pm \in \{-1,+1\} \tag{1.22}$$

Here we have denoted the two components of the Majorana spinor ψ^μ by ψ^μ_+ and ψ^μ_- which is suggested by the fact that after solving the eoms the first component only depends on $\tau+\sigma$ while the second only depends on $\tau-\sigma$. A similar freedom

[20] $\psi^+ \propto \psi^0 + \psi^1$ should not be confused with the spinor component ψ^μ_+ which will be introduced below.

[21] In calculating partition functions, a fermion-field is anti-periodic in time (if no further trace-insertion acts on this field in operator formalism). The modular group (which is a important symmetry in string theory) maps sectors in the partition that correspond to certain periodicities to other sectors. Thereby a spinor ψ that is periodic in σ and anti-periodic in time will get mapped to a different sector. This explains the presence of different boundary (or periodicity) conditions as well as the presence of the GSO-projection.

like in (1.22) exists also for open strings, where the supersymmetric partner of the bosonic boundary condition (eg. Neumann-type: $\partial_\sigma X^\mu = 0 \Rightarrow \partial_+ X^\mu = \partial_- X^\mu$, $\partial_\pm \equiv 1/2(\partial\tau \pm \partial_\sigma)$) becomes:[22]

$$\psi_+(\tau,\sigma) = \kappa(\sigma)\psi_-(\tau,\sigma) \qquad \text{for } \sigma \in \{0,\pi\},\ \kappa(\sigma) \in \{\pm 1\} \tag{1.23}$$

Depending on the sign κ one distinguishes between Ramond (R) ($\kappa = 1$) and Neveu-Schwarz (NS) ($\kappa = -1$) fermions. It turns out that unless one imposes a projection, the GSO-projection, the NS-sector contains a tachyon. The GSO-projection is also needed for modular invariance. Even though defined on the whole string-spectrum, the action of the GSO-projection on the R-ground-states is particularly interesting. Solving the equations of motion subjected to the boundary conditions, one discovers a zero-mode in each ψ^μ-coordinate. In light-cone-gauge one has therefore 8 zero-modes b^i which are anti-commuting and fulfill a Clifford-algebra:

$$\{b^i, b^j\} = \eta^{ij} \qquad i,j \in \{2 \ldots D-1\} \tag{1.24}$$

Thus one can represent the above algebra on a vector space with the following basis: $|s_1, s_2, s_3, s_4\rangle$ with $s_i = \pm\frac{1}{2}$. The resulting vector space can then be described as the sum of a vector space of positive chirality and another one of negative chirality. Performing the GSO-projection eliminates one chirality from the massless ground-state. In the closed string sector there are two sectors containing fermions depending on the combination of left- and right-moving sectors: These are the NSR and the RNS sectors, while the NSNS- and the RR-sector make up space-time bosons. In the open string there are just two sectors, the NS- and the R-sector, the latter containing the space-time fermions. As the GSO projection picks up one chirality, there is still the freedom to choose equal or opposite chiralities on left- and right-movers. Equal chiralities lead to the Type IIB superstring, while opposite chiralities yield Type IIA. Upon compactification on a circle, this does not make big difference, since both theories are then related by a perturbative duality, the so called *T-duality*. The massless spectrum of Type IIA theory can be found in table 1.1, its Type IIB pendant is given in table 3.1, page 62. While the resulting spectrum is supersymmetric, it is much harder to show that the interacting theory is supersymmetric as well. We will not investigate this topic.

It is possible to build modular-invariant partition functions that consist only of RR and NSNS sectors. The resulting theories are called Type 0A and Type 0B. They do not contain any fermions in the closed string sectors and are plagued with tachyons. However there exist interesting generalizations of these Type 0A/B by performing an *orientifold*[23] projection of these theories. This removes the closed-string tachyon and introduces fermions via a necessary open string sector (cf. [18] and references therein). There exist non-supersymmetric orientifolds of Type 0B that are completely tachyon free. Something similar is

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[22] There is still a redundancy in the following equations: By a field redefinition one can set $\kappa(0) = +1$.

[23] We will introduce orientifolds in section 1.6. In addition we devoted a whole chapter to these constructions (cf. chap. 3).

bosons	
NS-NS metric g_{ij}, 2-form B_{ij} dilaton ϕ	R-R vector A_i, 3-form C_{ijk}
fermions	
NSR gravitino $\psi_{i\dot{a}}$	RNS gravitino $\tilde{\psi}_{jb}$

Table 1.1: Massless closed-string spectra of Type IIA theory

known for the heterotic string as well: If one constructs the heterotic string in the NSR formalism one discovers that by changing the GSO-projection one can obtain a tachyon-free non-supersymmetric $O(16) \times O(16)$ string theory in ten space-time dimensions (cf. [19,20]). Several other non-supersymmetric modular-invariant variants of the heterotic string (which contain however tachyons) are known.

It is a natural task to consider $N > 1$ world-sheet supergravities. However it turns out that for $N = 2$ the critical space-time dimension would be 4 with a $(2, 2)$ space-time signature (which is phenomenologically uninteresting), while for $N = 4$ the dimension is even negative, and thus unacceptable for a reasonable space-time interpretation.

Space-time supersymmetry

Space-time supersymmetry is a desirable feature for physical theories. This has several reasons. The probably most important one is the *hierarchy problem*: In electroweak-theory the big difference between electroweak-scale (which is about 246 GeV, the *vacuum expectation value* (VEV) of the Standard Model Higgs field) and Planck-scale ($1.22 \cdot 10^{19}$ GeV) is believed to be very unnatural. Furthermore the parameters describing the Higgs-boson (which is the only scalar particle of the Standard Model) receive enormous contributions from radiative corrections up to the Planck scale. In order that these parameters take exactly those values required by measurements at typical "high-energy" experiments, the values have to be met within enormous precision (something like one part in 10^{30}) at the Planck scale. Furthermore this fine-tuning has to be repeated at each order of perturbation theory. In parallel the higher order corrections exceed in general the lower order approximations.

Grand unified theories

In (most) *grand unified theories* in general a second hierarchy problem comes along which is due to an additional Higgs particle. The underlying idea of grand unified theories is the following: Each lepton generation comes up with a quark-generation (or flavor) which however sits in a separate representation. One could now try to unify leptons with quarks in multiplets of the gauge group. This is achieved for example in the *Pati-Salam* (PS) $SU(4) \times SU(2)_{\mathrm{R}} \times SU(2)_L$-

model where the leptons correspond to a fourth color (cf. [21]). Each generation of matter transforms in a $(4,2,0)$ and $(4,0,2)$ representation of the gauge group. This Pati-Salam model has two interesting features, that are common to most other GUTs as well:

- Additional matter that is absent in the ("minimal") Standard Model (In Pati-Salam $SU(4) \times SU(2)_{\mathrm{R}} \times SU(2)_L$: right-handed neutrinos)
- The electric-charge is quantized

Quantization of electric-charge is in general true for models with simple gauge-group but also for this semi-simple example. In unifications with simple gauge-group the SM gauge-group is embedded into a larger, simple Lie-group G:

$$SU(3) \times SU(2) \times U(1)_Y \hookrightarrow G \tag{1.25}$$

Thus not only leptons and quarks become unified, but gauge-bosons of different gauge-groups as well. Well known examples for GUTs with simple gauge group G are $SU(5)$, $SO(32)$ and even $E(6)$ GUTs, the latter based on the exceptional group $E(6)$.[24] Among several interesting and attractive features of GUTs we want to mention the probably best known: GUTs in general predict *proton decay*. Proton decay, if present, can be measured (up to a certain bound) by experiments. Several GUTs have already been ruled out by experimental data. Supersymmetry suppresses the decay rate considerably. For example the non-supersymmetric $SU(5)$ GUT is forbidden, while its supersymmetric extension is still in accord with the bound given by current proton decay experiments. Analogous statements can be made for $SO(10)$.

Now we address a second hierarchy problem that comes along with most GUTs. What is important in GUTs, is that the unifying gauge-symmetry has to be broken at some scale, which is of course above the electro-weak scale. This will be done in general by some Higgs mechanism with the corresponding Higgs-field acquiring a VEV $\langle 0|\Phi|0\rangle = w$ which is of the order of the unification scale. We assume that the unification scale *a priori* does not coincide with the Planck scale. The running of the couplings strongly suggests that it is of the order of 10^{15} to 10^{16} GeV.[25] The second gauge-breaking is the usual electro-weak symmetry breaking which occurs at a VEV of $\langle 0|\phi_{\mathrm{e.w.}}|0\rangle = v \approx 246$ GeV. A generic Higgs potential looks like:[26]

$$V = -\frac{A}{2}\Phi^2 + \frac{B}{4}\Phi^4 - \frac{a}{2}\phi^2 + \frac{b}{4}\phi^4 + \frac{\lambda}{2}\Phi^2\phi^2 \tag{1.26}$$

The term proportional to λ is generic and thus has to be included. The GUT scale value is obtained, if we tune A and B such that: $w^2 = A/B$. The problem

[24] The $SU(5)$ model was proposed by Georgi and Glashow [22], the SO(32) theory by Georgi [23] in parallel to Fritzsch and Minkowski [24]. The $E(6)$ model was found by Gursey, Ramond and Sikivie [25].

[25] The first value is already excluded by experiment, and assuming solely the SM particle content will not lead to gauge coupling unification.

[26] We have suppressed group indices which are present since the Higgs fields transform under the gauge group.

occurs for the VEV of the second (i.e. the electroweak) Higgs: Since $v^2 = (a - \lambda w^2)/b$ has to be obeyed this requires a fine tuning of a to one part in 10^{26}. Radiative corrections will require this fine tuning at each order in perturbation theory. If present, supersymmetry ensures however that radiative corrections do not destroy the hierarchy and parameters do not have to be retuned. On the other hand supersymmetry has to be broken. Requiring the hierarchy to be preserved by this breaking leads to the prediction, that supersymmetric partners of the known particles should show up at 1 TeV.

Other mechanism like composite Higgs-particles have been proposed to circumvent the hierarchy problem without the use of supersymmetry. However these approaches are plagued with other difficulties.

Inspired from string-theory, it has been suggested that *extra large dimensions* could solve the hierarchy problem as well. In these scenarios the known gauge interactions are restricted to a lower dimensional subspace (a *brane*) while gravity propagates in the entire space (often denoted by "bulk"), which in most models has relatively large,[27] but compact directions. Future experiments can put severe constraints on the size of possible extra large dimensions, which might sustain or rule out these proposals.

As a third argument for supersymmetry, we mention the unification of the Standard Model couplings at a scale of 10^{16} GeV if one assumes the supersymmetry-breaking scale at about one TeV.

1.5 Compactifications

It goes back to the early twenties of the 20$^{\text{th}}$ century that *Kaluza* suggested a theory with an additional small dimension. Even though this dimension might not be discovered directly due to its smallness, it influences the four dimensional physics indirectly. As string theory on flat backgrounds has too many dimensions of unrestricted size, one has to figure out some explanation, why only four space-time dimensions are seen. A very fruitful idea is to compactify string theory on some tiny space $\mathcal{X}_d$:

$$\mathcal{M} = \mathbb{R}^{(1,D-d-1)} \times \mathcal{X}_d \tag{1.27}$$

By this we obtain effectively a theory with one time and $D - d - 1$ space dimensions. The exact form of the space $\mathcal{X}_d$ has big influence on the theory seen in uncompactified space. If we compactify a 10-dimensional $\mathcal{N} = 1$ superstring theory on a *Calabi-Yau* (CY) space, $\mathcal{N} = 1$ will be present in $10 - d$ dimensional space time.[28] Furthermore the chiral massless spectrum is determined by topological data of $\mathcal{X}_d$. The Calabi-Yau space admits in general additional structures like gauge-bundles. Physical requirements like anomaly-cancellation put further constraints on the geometry. The topic is too extended in order to

[27] By "large" we mean much bigger than the Planck length, and in order to solve the hierarchy problem: in the region up to a few TeV.

[28] To be precise, one has to deform the CY space to take the α'-string corrections to the supersymmetry algebra into account.

enter into details. For some geometric aspects of compactifications we refer the reader to the book of GSW [5].

A special case of compactification spaces are *orbifolds* to which we have devoted the next chapter. Roughly speaking, an orbifold is the orbit-space of some discrete group G that acts on a manifold $\mathcal{B}$:

$$\mathcal{X}_d = \mathcal{B}/G \tag{1.28}$$

The action of G may admit fixed-points, which usually result in singularities on $\mathcal{X}_d$. If the string theory on $\mathcal{B}$ is known, it is comparatively easy to construct the orbifold by G. Even though $\mathcal{X}_d$ might be singular in some points, string propagation turns out to be regular (in most cases). In all of our thesis we encounter either tori (that can also be interpreted as fixed-point free orbifolds) or toroidal orbifolds of $\mathbb{Z}_N$-groups or products thereof.

1.6 Open strings and unoriented string theories

We have already seen that the perturbative spectrum of the heterotic string leads to a non-abelian gauge-symmetry in the low-energy effective action. However both Type II theories do not show this gauge-symmetry. If one does not insist on 10-dimensional Lorentz-invariance, one can include gauge-symmetries in Type II theories as well. One way to achieve this is to include open strings, and in general: world-sheets with boundaries. One can assign charges to the end-points of open-strings in the way proposed by Chan and Paton (cf. [26]). In figure 1.5 we have depicted an open string with two charges n and $\bar{n}$ at its

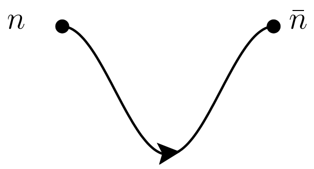


Figure 1.5: Open-string with Chan-Paton charges n and $\bar{n}$

endpoints. It can be shown that one can define consistently string perturbation theory if one assumes that both endpoints are represented by n-dimensional vectors in $\mathbb{C}^n$. The resulting string theory admits a global $U(n)$ symmetry, which is promoted to a local (i.e. gauge-) symmetry of the low-energy effective action. The $\bar{n}$ signals that the right-endpoint transforms in the complex conjugate representation with respect to the left one.

D-branes

The loci of open string endpoints can be associated to so called *Dp-branes*, where $p+1$ is the space-time dimension of these loci. In supersymmetric string-theories there exists a connection between Dp-branes and the supercharges which are preserved by these objects.[29] If several D-branes are present

[29]Not every possible locus (more precisely: submanifold) the Dp-brane wraps can be associated with supercharges. The submanifold has to fulfill additional condition, eg. the sLag condition.

they may or may not preserve some or several supercharges. The number of supercharges preserved by a D-brane configuration determines the amount of supersymmetry of the particular string model. Supersymmetric Dp-branes are usually *Bogomolnyi-Prasad-Sommerfield (BPS) states*, i.e. states with a reduced amount of supersymmetry that saturate the BPS-bound. BPS-states carry a *central charge* Z of the super-symmetry algebra which is a conserved charge. This fact made D-branes so important in the second "string revolution". String-theory was defined so far by a perturbative expansion, very similar to the way in which Feynman rules may be introduced by hand in electrodynamics. Whatever the correct non-perturbative definition of string-theory would be, it is extremely likely that it preserves the BPS-property, especially in the process of renormalization. On the other hand BPS-solutions were known to appear in the form of p-dimensional soliton-like solutions in the low-energy effective actions of string theories. Besides D-branes there exist other BPS-states in string theory as well. By identifying BPS states in perturbatively inequivalent theories the notion of an *M-theory* was born. M-theory is considered to be the unifying theory which includes all superstring theories as special limits of the M-theory moduli-space.[30] D-branes have also proven extremely useful in explaining Bekenstein-Hawking entropy at a microscopic level.[31]

Type I and orientifolds

We have claimed so far that Type II theories, if they contain open-strings, will break 10-dimensional Lorentz-symmetry. This is not disastrous, since for phenomenological reasons we will break this symmetry anyway at some point. So far we did not explain why we are not allowed to introduce freely some D9-branes in Type II, thereby maintaining Lorentz-invariance. It will become soon clear, that D-branes carry a special type of charge, a so called *Ramond-Ramond (RR) charge*, which is of topological type, and that this charge has to cancel in total. The RR-charge of a D-brane constitutes its central charge.

The low energy limit of Type I theory is a 10-dimensional $\mathcal{N} = 1$ supergravity coupled to 10-dimensional $\mathcal{N} = 1$ supersymmetric Yang-Mills with gauge group $SO(32)$. Both open and closed strings admit a further symmetry, which is world-sheet parity. World-sheet parity reverses the orientation of the world sheet, while leaving the action invariant. This has two effects:

- Only closed string-states that are invariant under the world-sheet parity Ω are kept in the spectrum.

- Due to the formula $\chi = 2 - 2h - b - c$ for the Euler-character χ we need to include the Klein-bottle ($h = 0$ handles, $b = 0$ boundaries, $c = 2$ cross-caps) as the second closed-string one-loop vacuum amplitude.

It turns out that the Klein-bottle amplitude has severe divergences. They are interpreted as uncanceled RR-charges under which the so called *orientifold plane*

[30] 11-dimensional supergravity is another limit in the M-theory moduli-space.
[31] At least for some supersymmetric black hole configurations.

(O-plane) is charged. (The O-plane corresponds to the cross-caps in the Klein-bottle). In analogy to field theory these divergences are called RR-*tadpoles*. As D-branes carry RR-charges as well they may serve as a neutralizer of the O-plane charge, provided that their charge has the right sign and value. This is indeed the case. In Type I the RR-charge is exactly canceled by 32 D9-branes. In computing the open string partition function we have to include the parity projection Ω as well. This implies that we have to introduce the *Möbius-strip* ($b = 1$, $c = 1$) besides the cylinder ($b = 2$, $c = 0$). The projection together with the RR-tadpole cancellation conditions implies that the $U(32)$-symmetry gets broken to $SO(32)$. The only gauge groups which can be obtained in the perturbative spectrum of Type I and compactifications thereof are orthogonal, symplectic and (under certain circumstances) unitary groups.

The Type I construction can be generalized. On one hand Type I can be compactified on some space $\mathcal{X}_d$. As before $\mathcal{X}_d$ might be an orbifold. We can also gauge a combination $s\Omega$, where s acts on space-time such that $s\Omega$ is a symmetry of the string theory under consideration. We can even try to include several such elements. However we will show in section 3 that this does not lead to new consistent models in most cases. Given a projection via $s\Omega$ and a compactification space $\mathcal{X}_d$ there might be several inequivalent ways to cancel the RR-tadpole of the O-plane(s).[32] All these generalizations which include the world sheet-parity in some way are summarized by the term: "orientifold".

1.7 Chiral fermions in open string theories

As half of the thesis deals with chiral fermions from the open string sector in one way or the other, we want to make some comments here. Chiral-fermions are an essential feature of the SM. In string theory they can be obtained in many ways (cf. the introduction to chap. 6, p. 139). For open string theories three mechanism are very prominent:

1. Open strings with endpoints on D-branes with non-trivial topological intersection number
2. Open strings with endpoints on D-branes which carry different magnetic background fields
3. Open strings stuck to a singularity

For flat space-time, the first method was discovered in [27]. The second method was (to our knowledge) first applied to model building in [28]. Both methods are related by T-duality, if the branes intersect as lines when restricted to a T^2. T-duality acts on each T^2 in *one* coordinate by $R/\sqrt{\alpha'} \to \sqrt{\alpha'}/R$. The classical solutions for both scenarios are depicted in figure 1.6 and 1.7. (These figures show the string, its boundaries and the world sheet, as well as the D-branes for the intersecting scenario.) The quantized version has some features in common with the classical solution. In the case of intersecting D-branes (fig. 1.6), one

[32] In more complicated spaces $\mathcal{X}_d$ the O-planes consist of several parts. Therefore, we refer to "several O-planes".

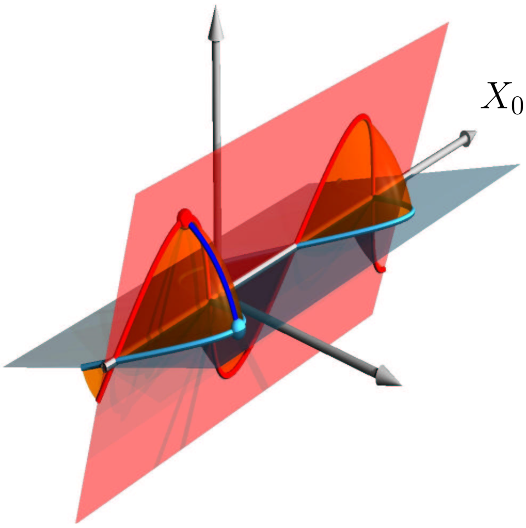


Figure 1.6: Time evolution of an open-string with endpoints located on D-branes intersecting at an angle. The classical string oscillates around the intersection point. Upon toroidal compactification on a T^2 angled D-branes are T-dual to magnetic backgrounds (right figure).

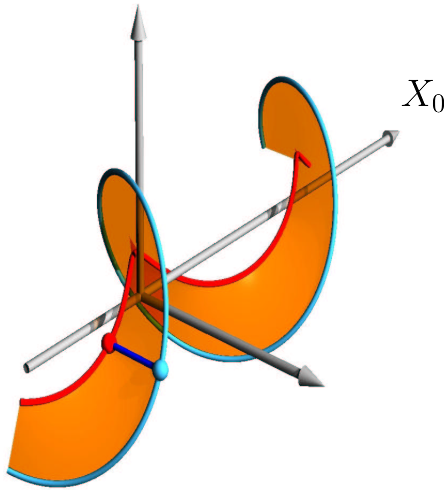


Figure 1.7: Time evolution of a bosonic open-string in constant magnetic background fields. The classical string rotates around a point, whose position is however *not* determined by the NS-fields on its boundaries.

The string is a blue line, while the world-sheet boundaries are in red and "skyblue". The string depicted obeys the classical eoms. Its lowest (non-zero) mode is excited. In fig. 1.6 the D-branes are drawn in transparent colors, while in fig. 1.7 the branes are two-dimensional. The world sheet is in transp. orange.

sees that the string oscillates around the intersection point. The string which is coupled to the magnetized D-branes circulates around some point as well. This point is classically not restricted. In the quantized version it corresponds to a Landau level. The infinite Landau degeneracy gets finite after compactification, e.g. compactification on a torus. The easiest way to see the appearance of chiral fermions is first to note that by the altered boundary conditions the number of Ramond-zero-modes b^i (cf. section 1.4) is reduced, such that (for suitable boundary condition) only one Ramond-state survives:

$$\begin{array}{ccc} \text{homogenous} & & \text{inhomogenous} \\ \text{boundary conditions} & & \text{boundary conditions} \\ \big|s_1, s_2, s_3, s_4\rangle\big|_{\text{GSO-proj.}} & \longrightarrow & \big|+\tfrac{1}{2}\rangle \end{array} \tag{1.29}$$

By "homogenous" we mean that there are identical boundary conditions on the left- and right-endpoint of the string (at least concerning the derivatives). Each chiral fermion obtained this way appears with a multiplicity that is determined by the bosonic zero-modes, where the sign has to be properly taken into account. This multiplicity is the topological intersection number or the Lan-

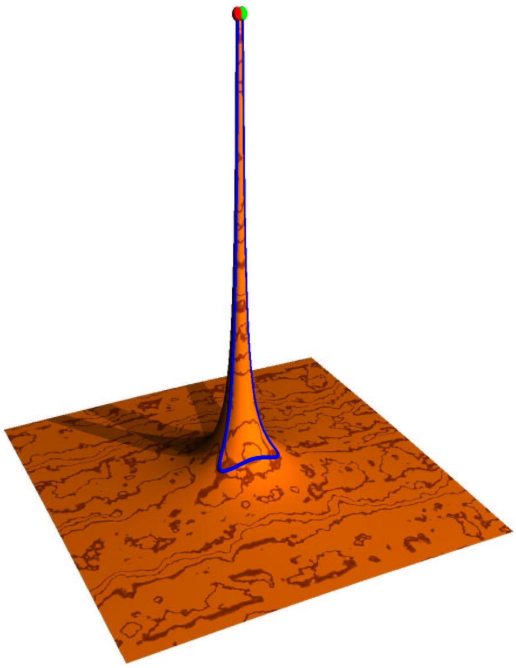


Figure 1.8: Open-string located at a singularity (schematic): A D-brane which is bound to a singularity in compactification space is the locus of open-string end-points. (The string is painted in blue, its endpoints are red and green). Open superstrings in this sector might admit chiral fermions. If the string end-points belong to different stacks of D-branes (denoted by a and b), this can lead to chiral fermions in bifundamental representation $(n_a, \bar{n}_b)$ of the associated gauge group $U(n_a) \times U(n_b)$.

dau degeneracy which might be calculate via the *Atiyah-Singer index theorem* for twisted spin-complexes. Lower dimensional D-branes which have exactly half the dimension of the embedding space like in figure 1.6 are encountered in $\bar{\sigma}\Omega$-orientifolds. In these orientifold $\bar{\sigma}$ acts as complex conjugation on each T^2. (The compactification space is a product of T^2's or an orbifold thereof.) We use $\bar{\sigma}\Omega$-orientifold constructions in chapter 6 and 7 and obtain interesting chiral spectra. In chapter 6 we alternatively consider the T-dual magnetized situation as well. Even though we do not apply it in this thesis, we want to mention that chiral fermions can be obtained from D-branes which are located at singularities (cf. [29, 30]). The orbifold case, especially the $T^2/\mathbb{Z}_3$ case has been exhaustively explored (cf. [29,31] and references therein). A D-brane that is stuck to an (orbifold) singularity is called a *fractional brane*. This is due to the fact that it carries only a fractional amount of untwisted RR-charge in comparison to an ordinary D-brane. Open string states in the Ramond-sector are described by:[33]

$$|s_1, s_2, s_3, s_4; \Lambda_{ij}\rangle\big|_{\text{GSO-proj.}} \tag{1.30}$$

Λ_{ij} is the Chan-Paton matrix that encodes the CP-dofs. In symmetrizing the state with respect to an orbifold group G one encounters conditions like:

$$|g(s_1, s_2, s_3, s_4); (g^{-1}\Lambda g)_{ij}\rangle\big|_{\text{GSO-proj.}} = |s_1, s_2, s_3, s_4; \Lambda_{ij}\rangle\big|_{\text{GSO-proj.}} \qquad \forall g \in G \tag{1.31}$$

This will in general reduce the number of CP-dofs such that the gauge group gets broken. In addition many zero-modes $|s_1, s_2, s_3, s_4\rangle$ will be projected out. Depending on the orbifold group G this can result in a single chiral fermion.

[33] For untwisted boundary conditions.

We have illustrated the situation where an open string is stuck to a singularity in figure 1.8.[34]

[34]Of course such a singularity can not exist in one dimension. In order to get chiral fermions in four space-time dimensions, the singularity has to be complex three-dimensional.

Chapter 2

Orbifolds

In this chapter we will introduce the notion of a (string-theoretic) orbifold. While we give several refenrences during this chapter, the fundamental publications for string-theoretic orbifolds are the two papers by Dixon, Harvey, Vafa and Witten [32, 33].

We will present *orientifolds* in the next chapter. Orientifolds are string theories with an orbifold group containing elements which interchange left- and right-moving sectors of the theory.

It is assumed that the reader is familiar with the basic concepts of string theory. After giving the ideas of orbifold constructions we will present the d dimensional torus T^d as a concrete example. The formally introduced orbifold torsion can be identified with the exponential of closed NSNS two-form fluxes $B_{\mu\nu}$ in this case. We will explain the T-duality group $SO(2,2,\mathbb{Z})$ for the two-torus, because we will use these results in later chapters. A good review for T-duality is the report by Giveon, Porrati and Rabinovici [34]. Finally, we will present the asymmetric $T^4/\mathbb{Z}_3^L \times \mathbb{Z}_3^R$ orbifold in some detail. This introductory chapter on orbifolds is far from being exhaustive. Even though the notion of an orbifold is introduced, it is impossible to enter into the details. This chapter is meant as a tool to understand the concrete models which are presented in the following chapters, especially chapter 5 and 7.

2.1 General construction of orbifolds

Compactifications in superstring theory are usually of the form:

$$\mathcal{M} = \mathbb{R}^{(1,9-d)} \times \mathcal{X}_d \tag{2.1}$$

Whereas $\mathbb{R}^{(1,9-d)}$ is the flat Minkowski space, $\mathcal{X}_d$ is a small d dimensional, compact space. Even though an orbifold can be of any dimension we will concentrate on the dimension $d \leq 6$ case since it seems to be the most relevant one for superstring compactifications to $10-d$ dimensional space-time. In common orbifold constructions $\mathcal{X}_d$ is obtained as a quotient of a manifold $\mathcal{B}$ by a group G acting in a discrete way on $\mathcal{B}$:

$$\mathcal{X}_d = \mathcal{B}/G \tag{2.2}$$

String theory is usually defined on spaces admitting a metric. Especially this is the case for $\mathcal{B}$. As the metric appears in several quantities like the Hamiltonian it is an essential structure of the theory. We require it to be invariant under the action of G. In order for G to be a symmetry of the theory on $\mathcal{B}$ we require that all physical quantities like transition amplitudes and especially the Hamiltonian stay invariant under G.

As G can admit fixed-points in $\mathcal{B}$ (more generally: fixed-sets) the orbifold might get singular at these points. In going from the geometrical space to string theory one is especially interested in the Hilbert space of the string theory living on $\mathcal{X}_d$, or more precisely, in the Hilbert space associated with $\mathcal{M}$ in (2.1) (We call this Hilbert space $\mathcal{H}_{\mathcal{X}}$). The states in $\mathcal{H}_{\mathcal{X}}$ can partially be obtained by projecting on the G-invariant subspace of $\mathcal{H}_{\mathcal{B}}$ ($\mathcal{H}_{\mathcal{B}}$ the Hilbert space of the string-theory on $\mathcal{B}$). It turns out that there are additional states in the Hilbert space coming from so called *twisted sectors* $\mathcal{H}_g$, $g \in G$ which form subspaces of $\mathcal{H}_{\mathcal{X}}$. These states stem from closed-strings which are closed on $\mathcal{X}$ but on $\mathcal{B}$ only by an element g of G:

$$X(\tau, \sigma + 1) = gX(\tau, \sigma), \quad g \in G \tag{2.3}$$

If the general solution of the equations of motion (eoms) for X on the space $\mathcal{B}$ is known, it is often quite easy to implement the modified (i.e. twisted) boundary (or: periodicity) condition (2.3). After quantizing the fields in this new sector, one can construct the resulting Hilbert space by known methods. Especially, one has to ensure that the states in the $\mathcal{H}_g$ sectors are invariant under all $h \in G$. A more detailed investigation reveals that states in $\mathcal{H}_g$ have to be invariant only under the centralizer C of g ($h \in C \Leftrightarrow hg = gh$). As the information on the particle spectrum is encoded in the partition function, this quantity is extremely important. The perturbative spectrum is encoded in the one-loop partition function. The latter enters the one-loop vacuum amplitude as an integrand. For closed oriented strings the one-loop amplitude is the *torus amplitude*.

The torus amplitude can be written as a path-integral with integration over fields of definite periodicity. Equivalently it can be calculated in the operator formalism as a trace over states corresponding to these periodicities in the σ direction and trace insertions corresponding to the τ (world sheet time) direction.[1] The torus amplitude (including the world sheet fermions) takes then the form:

$$T = V_{10} \int_{\mathcal{F}} \frac{d^2\tau}{4\mathfrak{Im}\tau} \int \frac{d^{10}p}{(2\pi)^{10}} \operatorname{Tr} \mathbf{P}_{\mathrm{GSO}} (-1)^F q^H \bar{q}^{\tilde{H}} \tag{2.4}$$

τ is the modular parameter of the torus, $q = \exp(i2\pi\tau)$ and $\mathcal{F}$ is one fundamental region of the torus. V_{10} denotes the ten dimensional regulated space time volume and $\mathbf{P}_{\mathrm{GSO}} = \frac{1}{2}\big(1 + (-1)^f\big)$ the GSO projection (f the world sheet fermion number). F is the space time Fermion number ($(-1)^F = -1$ in the RNS and NSR sector, otherwise $= 1$). The trace in (2.4) is over the

[1] No trace insertion corresponds to periodicity of bosonic fields in the time (or: τ) direction and *anti*-periodicity of fermionic fields in this direction.

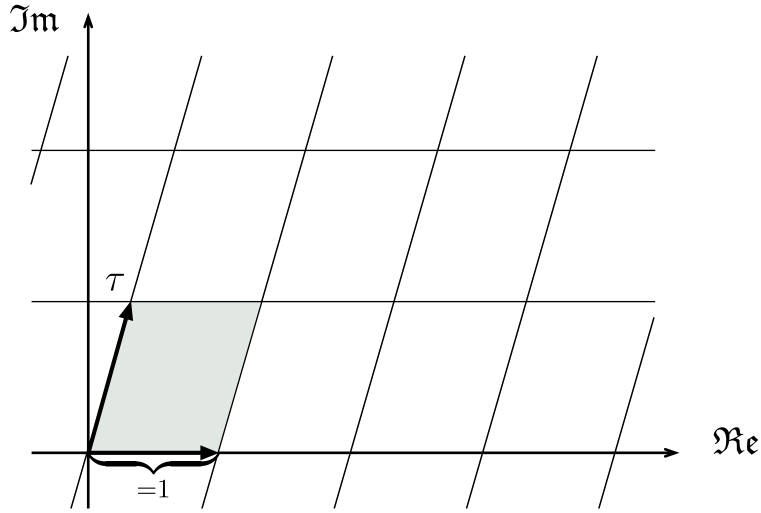


Figure 2.1: The Torus lattice Γ with complex structure τ. One fundamental region of the two-torus is shaded.

world sheet bosonic and over the fermionic sector. The fermionic sector divides into a Neveu-Schwarz (NS) sector (corresponding to world sheet fermions anti-periodic in σ) and a Ramond (R) sector (corresponding to periodic fermions). Integrating in (2.4) over p^0 and one component of momentum p^i (which are part of $H + \widetilde{H}$) leads to an additional factor $(\alpha' \mathfrak{Im}\tau)^{-1}$.[2] We notice that the resulting measure $\left(4\alpha'(\mathfrak{Im}\tau)^2\right)^{-1} d^2\tau$ is invariant under modular transformations $\tau \to \frac{a\tau+b}{c\tau+d}$ with $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL(2,\mathbb{Z})$. The torus is defined as the orbit space of a two dimensional lattice acting additively on $\mathbb{C} \simeq \mathbb{R}^2$ (cf. figure 2.1):

$$T^2_\tau \equiv \mathbb{C}/\Gamma \qquad\qquad \Gamma_\tau = \{m + n\cdot\tau \,|\, m,n \in \mathbb{Z}\} \tag{2.5}$$

$g \in SL(2,\mathbb{Z})$ acts on τ as described above, or equivalently on a vector $\vec{v} = (m,n)^T \in \Gamma$ as matrix multiplication from the left by a matrix described above. It is therefore obvious that Γ and as a consequence T^2_τ is invariant under $SL(2,\mathbb{Z})$. This modular invariance should also be reflected in the torus partition function. The torus amplitude is modular invariant if the integrand of the $d^2\tau$-integration (the trace including the remaining momentum integral) is modular invariant. Since the integrand is essentially the partition function, its modular invariance is commonly referred to as *modular invariance* of the partition function. With the explicit modular invariance of the integrand one is free to choose a fixed fundamental region $\mathcal{F}$ which under the action of G is mapped to the complete upper half plane $\mathbb{H}^+$. The choice $\mathcal{F}_0 = \{|\tau| > 1,\, |\mathfrak{Re}\tau| < 1/2,\, \mathfrak{Im}\tau > 0\}$ (cf. fig. 2.2) eliminates explicitly potential divergencies in the region $\tau \to 0$. This is in contrast to field theory where this limit corresponds to UV-divergencies. Therefore modular invariance is essential for the finiteness of string theory. Anomalies in field theory have several interpretations. They signal a breakdown of classical symmetries at the quantum level. Gauge symmetries in field

[2] To regularize the momentum integral one has to perform a Wick rotation.

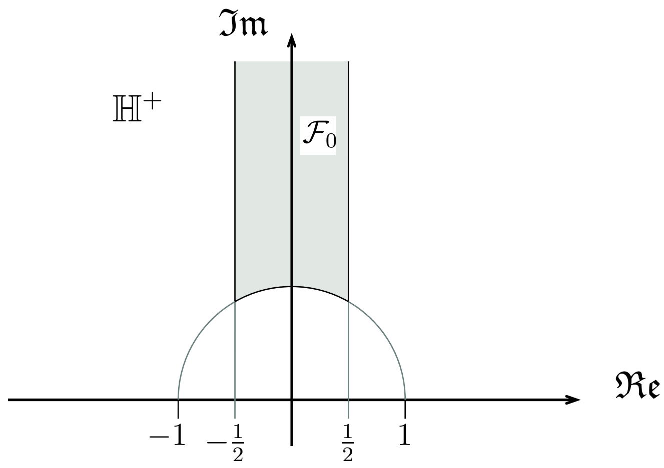


Figure 2.2: The upper half plane $\mathbb{H}^+$ and the fundamental region $\mathcal{F}_0$ of the complex structure.

theory play an important role in decoupling unphysical states in physical quantities like transition amplitudes. It can be shown that in order to decouple unphysical states (i.e. unphysical vertex operators) in string theory, modular invariance is needed. It ensures also the absence of anomalies in the low energy effective field theory limit of the corresponding string theory. In field theory anomalies in gauge symmetries ruin the renormalizibility of a theory. Therefore modular invariance in string theory is intimately connected to the finiteness of the theory, the absence of anomalies and the decoupling of unphysical states.

In orbifolds the torus vacuum amplitude (2.4) gets modified. One has to sum up the traces over sectors $\mathcal{H}_g$ representing states with g-twisted boundary conditions (2.3) in the σ-direction. One will also have to insert projectors of the form

$$\frac{1}{|G|}\sum_{h\in G} h \tag{2.6}$$

in the trace over states in $\mathcal{H}_g$, thereby projecting onto G-invariant states. An insertion of h into the trace corresponds to integrating over fields with periodicity

$$X(\tau+1,\sigma) = hX(\tau,\sigma) \tag{2.7}$$

in the path-integral formalism. This only makes sense for h and g commuting (i.e. $hg = gh$). Commonly a sector of the partition function that corresponds to fields with periodicity (2.3) and (2.7) is represented by:

$$h\underset{g}{\square} \tag{2.8}$$

A modular transformation of the parameter $\tau \to \frac{a\tau+b}{c\tau+d}$ has the same effect as transforming

$$h\underset{g}{\square} \to h^a g^b \underset{h^c g^d}{\square}, \qquad \text{for } hg = gh \tag{2.9}$$

with τ unchanged. If one knows all trace insertions in the σ untwisted sectors, i.e. all contributions $h\underset{1}{\square}$ one can construct a big part of the twisted sectors $h^n\underset{h^m}{\square}$ by applying eq. (2.9). This proves extremely useful in the construction of so called *left-right asymmetric orbifolds*, where solving of the boundary conditions (2.3) is not problematic. This is due to the fact that the asymmetric action on the center of mass (com.) coordinate of the string is not well defined. World sheet fermions which correspond to anti-commuting fields come in four different types for each h, $g \in G$: The NS fermions have an additional twist of -1 in the σ direction and the $(-1)^f$ trace insertion of the GSO projection corresponds to an additional twist of -1 in the world sheet time direction that is also denoted by τ. The whole one-loop partition function of the closed-string sector is then given by:

$$Z_{\mathcal{X}}(q,\bar{q}) = \frac{1}{|G|} \sum_{\substack{h,g\in G \\ hg=gh}} h\underset{g}{\square} \tag{2.10}$$

Although this expression is formally modular invariant there are some subtleties. They appear especially in so called asymmetric orbifolds. Eq. (2.2) defines an orbifold as a geometric space. Typically a string theory admits more symmetries than the background on which the string propagates. As the string splits into left- and right-moving parts, a symmetry can interchange these parts (like in orientifolds) or act differently on left- and right-movers. A generalized orbifold group might also contain elements which do both. It has been observed that in asymmetric orbifolds the naive partition function (2.10) might be ill defined. If certain conditions are not fulfilled[3] a sector which should be transformed to an equivalent sector by some element of the modular group, might however gain a non-trivial phase. (The modular group is represented only projectively on the partition function.) One remark is in order: If one wants to extract the spectrum from $Z(q,\bar{q})$ one still has to impose the condition that the number of left-moving excitations equals the number of right-moving excitations ($N = \widetilde{N}$). If G is a product of groups, i.e. $G = G_1 \times G_2$, $Z_{\mathcal{X}}$ in eq. (2.10) is not the only modular invariant partition function. In general $Z_{\mathcal{X}}$ can split into sectors which are not related by modular transformations. These sectors are allowed to aquire $U(1)$ phases $\epsilon(g,h)$ where g is the twist in the τ-, h the twist in the σ-direction. However higher loop consistency of string theory puts further constraints (cf. [35]) (in form of co-cycle conditions) on $\epsilon(c,d)$:

$$\epsilon(g_1 g_2, h) = \epsilon(g_1,h)\epsilon(g_2,h) \tag{2.11}$$

$$\epsilon(g,h) = \epsilon(h,g)^{-1} \tag{2.12}$$

$$\epsilon(g,g) = 1 \tag{2.13}$$

Imposing $\epsilon(h,g)$ to be invariant under the modular transformation $\epsilon(h,g) \rightarrow \epsilon(h^a g^b, h^c g^d)$ the sectors $h\underset{g}{\square}$ in the partition function (2.10) get multiplied by $\epsilon(h,g)$. The phase ϵ is commonly called *discrete torsion*.

[3]These obstructions are described for instance in [35].

We will now consider toroidal compactifications which are the starting manifold $\mathcal{B}$ (eq. (2.2)) for so called toroidal orbifolds. Toroidal orbifolds and compactifications play an essential role in chapters 5-7.

2.2 Torus compactification as an orbifold

If one wants to build orbifolds which descent form string theory on flat ten dimensional Minkowski space (with constant NSNS two-from potential B), where the string can be explicitly quantized, G is allowed to be a subgroup of the euclidian group acting on the base space $\mathcal{B} = \mathbb{R}^d$ (cf. eq. (2.1)). This group leaves the Hamiltonian of the string theory on $\mathcal{B}$ invariant. As the Hamiltonian on $\mathcal{B}$ splits into independent left- and right-moving parts (in light cone gauge: $H_{\text{bos.}\,,\,\text{L/R}} = \|\partial_\pm X\|^2_{\text{transv.}} + a_{\text{L/R}}$) one can mod out independent subgroups of $\text{Euc}(\mathbb{R}^d)$ in the left- and right-moving sector of the theory. However in constructing the twisted sector of an asymmetric orbifold one faces the question what the fixed points are. We will discuss this topic in section 2.3. As toroidal orbifolds (i.e. $\mathcal{X}_d = T^d/G$ with T^d a flat d dimensional torus) are of special interest, we will first consider strings living on

$$\mathcal{M} = \mathbb{R}^{(1,9-d)} \times T^d \tag{2.14}$$

The bosonic string action on this background has the form:

$$S_{\text{bos}} = -\frac{1}{4\pi\alpha'}\int_{\mathcal{M}} d^2\sigma \left(\partial_\alpha X^\mu \partial^\alpha X_\mu - B_{\mu\nu}\epsilon^{\alpha\beta}\partial_\alpha X^\mu \partial_\beta X^\nu\right) \tag{2.15}$$

We note that the equations of motion of a closed-string are not affected by a constant B-field as the variation of the action w.r.t. X^μ only contributes a boundary piece which is assumed to vanish. (The $\tau \to \pm\infty$ pieces of the boundary for an infinite cylinder are assumed to give no contribution. Quantizing the closed-string on a torus circumvents this problem because the torus has no boundary.) However the canonical momentum gets modified by the constant B-field:

$$P^\mu(\tau,\sigma) = \frac{\partial}{\partial \dot{X}_\mu} L(X, \partial X) = \frac{1}{2\pi\alpha'}\left(\dot{X}^\mu + B^\mu{}_\nu X'^\nu\right) \tag{2.16}$$

The integrated momentum $\int d\sigma P^\mu$ is τ-independent which follows from the equation of motion and the fact that $\partial\mathcal{M} = \emptyset$. Since

$$\begin{aligned} T^d &= \mathbb{R}^d/2\pi\Gamma \\ \Gamma &= \left\{n^i e_i \middle|\, n^i \in \mathbb{Z},\ e_i,\ i = 1\ldots d \text{ a fixed basis of } \mathbb{R}^d\right\} \end{aligned} \tag{2.17}$$

the torus is an orbifold itself, with a discrete, but fix-point free acting group $G = \Gamma \simeq \mathbb{Z}^n$. As Γ acts additively the $g = n^i$ twisted sector is characterized as follows:[4, 5]

$$X(\tau, \sigma + 2\pi) = 2\pi n^i e_i + X(\tau, \sigma) \tag{2.18}$$

---

[4] X, e_i, P and Π should be understood as vectors, not just as numbers.

[5] Here we set the closed-string length to 2π in world sheet coordinates, because this choice seems to be in common use for concrete mode expansions. The periodicities in formulæ like (2.3), (2.7) then changes from 1 to 2π.

This sector is usually called the *winding sector* since a string in a twisted sector with $g = n^i$ corresponds to a string winding n^i times around the i^{th} one-cycle. A solution in flat Euclidian space for the d $X^j(\tau,\sigma)$ coordinates involved in (2.18) has an oscillator part which is unchanged compared to the 2π periodic sector. In addition a piece linear in σ is needed to accomplish equation (2.18). Altogether the bosonic string coordinate in the n^i-twisted sector takes the following form:

$$X(\tau,\sigma) = x + \sqrt{2\alpha'}p\cdot\tau + n^i e_i\cdot\sigma + i\sqrt{\frac{\alpha'}{2}}\sum_{k,\in\mathbb{Z}^*}\left(\frac{\alpha_k}{k}e^{ik(\sigma-\tau)} + \frac{\tilde{\alpha}_k}{k}e^{-ik(\sigma+\tau)}\right) \tag{2.19}$$

Whereas the separation into left- and right-movers for the oscillators in the above expression is completely obvious, we will have a closer look at the parts linear in σ and τ. The condition that momentum states are invariant under s^i, i.e. [6]

$$s^i\,|p\rangle = |p\rangle \tag{2.20}$$

puts constraints on the allowed spectrum for p. With the momentum (2.16) one finds the following commutators for the modes in (2.19):[7]

$$[x^i, p^j] = i\sqrt{\alpha'/2}G^{ij} \qquad [\alpha^i_l, \alpha^j_m] = [\tilde{\alpha}^i_l, \tilde{\alpha}^j_m] = l\cdot\delta_{l+m,0}G^{ij} \tag{2.21}$$

Since[8] $i\Pi_j \equiv \int d\sigma P_j = i\big(\sqrt{2/\alpha'}G_{jk}p^k + (1/\alpha')B_{jk}n^k\big)$ is the generator of translations in the e_j direction, s^j acts as:[9]

$$s^j\,|p\rangle = \exp\big(i2\pi s^j\Pi_j\big)\,|p\rangle \tag{2.22}$$

Therefore s^j-invariant states have to fulfill:[10]

$$\vec{s}\cdot\Pi \in \Gamma^*, \quad \Gamma^* = \left\{m_i e^i |\, m_i\in\mathbb{Z},\ e^i e_j = \delta^i{}_j,\ i,j = 1\ldots d\right\} \tag{2.23}$$

Γ^* is the lattice dual to Γ in the sense that its elements $\vec{v}^*$ have integer scalar products with vectors $\vec{w}\in\Gamma$ and that the fundamental cells have inverse volumes. Invariance of the states under the full translation group Γ requires of course that $\Pi\in\Gamma^*$. With this information the p's in the n^i-twisted and s^j-invariant sector of eq. (2.19) can be expressed as:

$$p(\vec{m})_{\vec{s},\vec{n}} = e_k p^k = e_k\sqrt{\frac{\alpha'}{2}}\Big(G^{kj}\frac{m_j}{s^j} - \frac{B^k{}_i}{\alpha'}n^i\Big), \qquad \vec{m}\in\mathbb{Z}^d \tag{2.24}$$

The Hamiltonian of the theory is not explicitly B dependent when expressed in terms of $\dot{X}$ and X', or equivalently in terms of $\partial_\pm X$:

$$H_{\text{bos}} = \frac{1}{4\pi\alpha'}\int d\sigma\,(\dot{X}^2 + X'^2) = \frac{1}{2\pi\alpha'}\int d\sigma\,\Big(\big(\partial_+X\big)^2 + \big(\partial_-X\big)^2\Big) \tag{2.25}$$

---

[6] By acting with s^j we mean a translation by $2\pi s^j e_j$. This condition puts no constraints on the oscillator part of the state (both the bosonic and fermionic) since n^i acts trivially on the oscillators.

[7] G^{ij} is the metric of the torus dual to T^d.

[8] $B_{ij} \equiv e^\mu_i B_{\mu\nu} e^\nu_j$

[9] The metric G_{ij} of T^d which is dual to G^{ij} satisfies: $G^{ij}G_{jk} = \delta^i_k$.

[10] The following product $\vec{s}\cdot\Pi$ is defined by $(\vec{s}\cdot\Pi)^\mu \equiv \sum_j s^j\Pi^j e^\mu_j$ and $\Pi_j = G_{jk}\Pi^k$.

where we have defined:

$$\partial_\pm \equiv \tfrac{1}{2}\big(\partial_\tau \pm \partial_\sigma\big) \tag{2.26}$$

The bosonic oscillator part of H is:[11]

$$\begin{aligned} H_{\text{bos, osc}} = H + \widetilde{H} &= \frac{1}{2}\sum_{k\in\mathbb{Z}^*}\left(\alpha_{-k}\cdot\alpha_k + \tilde{\alpha}_{-k}\cdot\tilde{\alpha}_k\right) \\ &= \sum_{k\in\mathbb{N}^*}\left(\alpha_{-k}\cdot\alpha_k - a_{\text{bos}} + \tilde{\alpha}_{-k}\cdot\tilde{\alpha}_k - \tilde{a}_{\text{bos}}\right) \end{aligned} \tag{2.27}$$

with a the normal ordering constant which is $1/24$ for each transverse bosonic coordinate in light cone gauge. The oscillator part of the Hamiltonian is not affected by a constant B. It is the same as in the non-compactified theory. Requiring that the states $|p\rangle$ to be invariant under the whole lattice group Γ restricts the s^j in eq. (2.24): $s^j = 1\,\forall j$. We call the $\vec{m}$ excitations the *Kaluza Klein* (KK) modes because they correspond to the quantized momentum modes of a point particle compactified on the torus T^d. In contrast to the oscillator part, the part linear in τ and σ gets affected by the torus compactification. We will call it the *lattice part* since it depends on Γ. With the definition

$$\begin{aligned} p(\vec{m},\vec{n})_\pm &\equiv \frac{1}{\sqrt{2}}p(\vec{m})_{s^j=1,\vec{n}} \pm \frac{n^i e_i}{2\sqrt{\alpha'}} \\ &= \frac{e^k}{2}\left(\sqrt{\alpha'}m_k + \frac{1}{\sqrt{\alpha'}}\left(\pm G - B\right)_{kj}n^j\right) \end{aligned} \tag{2.28}$$

the lattice part of the bosonic field can be rewritten:

$$X_\Gamma(\tau,\sigma) = x + \sqrt{\alpha'}\big(p_+\cdot(\tau+\sigma) + p_-(\tau-\sigma)\big) \tag{2.29}$$

The lattice Hamiltonian H_Γ takes the form:

$$\begin{aligned} H_\Gamma &= p(\vec{m},\vec{n})_+{}^2 + p(\vec{m},\vec{n})_-{}^2 \\ &= \frac{1}{2}\left(m_i, n^j\right)\begin{pmatrix} \alpha' G^{ik} & -B^i{}_l \\ B_j{}^k & \frac{1}{\alpha'}\left(G - B^2\right)_{jl}\end{pmatrix}\begin{pmatrix} m_k \\ n^l\end{pmatrix} \end{aligned} \tag{2.30}$$

and

$$p(\vec{m},\vec{n})_\pm{}^2 = \tfrac{1}{4}\left(m_i, n^j\right)\begin{pmatrix} \alpha' G^{ik} & \left(\pm G - B\right)^i{}_l \\ \left(\pm G + B\right)_j{}^k & \frac{1}{\alpha'}\left(G - B^2\right)_{jl}\end{pmatrix}\begin{pmatrix} m_k \\ n^l\end{pmatrix} \tag{2.31}$$

As the splitting of the linear part of $P(\tau,\sigma)$ into $p(\vec{m},\vec{n})_\pm$ is unambiguous we can embed naturally $p(\vec{m},\vec{n})$ into a $2d$ dimensional lattice $\Gamma^{(d,d)}$ by the map:[12]

$$\begin{array}{ccc} \Gamma^d \times \Gamma^{d\,*} & \xrightarrow{\Upsilon} & \Gamma^{(d,d)} \\ \left(n^i e_i, m_j e^j\right) & \longmapsto & \big(p(\vec{m},\vec{n})_+, p(\vec{m},\vec{n})_-\big) \end{array} \tag{2.32}$$

[11] The dot product is meant to be the product w.r.t. the (dual) metric G_{ij}

[12] The product structure $\Gamma^d \times \Gamma^{d\,*}$ should be understood set theoretically. It is not of physical relevance because the Hamiltonian (2.30) couples vectors in both lattices.

Υ has the following matrix representation:

$$\Upsilon = \frac{1}{2}\begin{pmatrix} \sqrt{\alpha'}G_k{}^i & \frac{1}{\sqrt{\alpha'}}\big(G-B\big)_{kj} \\ \sqrt{\alpha'}G_l{}^i & -\frac{1}{\sqrt{\alpha'}}\big(G+B\big)_{kj} \end{pmatrix} \tag{2.33}$$

For later use in asymmetric orbifolds we calculate the inverse of Υ:

$$\begin{pmatrix} m_i \\ n^j \end{pmatrix} = \Upsilon^{-1}\begin{pmatrix} (p_+) \\ (p_-) \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{\alpha'}}\big(G+B\big)_i{}^k & \frac{1}{\sqrt{\alpha'}}\big(G-B\big)_i{}^l \\ \sqrt{\alpha'}G^{jk} & -\sqrt{\alpha'}G^{jl} \end{pmatrix}\begin{pmatrix} (p_+)_k \\ (p_-)_l \end{pmatrix} \tag{2.34}$$

Besides a positive definite and non degenerate quadratic form given by the Hamiltonian (2.30) we can define another non degenerate but non definite quadratic form of signature (d,d) which is of physical importance:

$$\frac{1}{2}\Xi\big(p(\vec{n},\vec{m})\big) \equiv H_{\Gamma+} - H_{\Gamma-} = \big(m_i,\, n^j\big)\begin{pmatrix} 0 & \frac{1}{2}\delta^i{}_l \\ \frac{1}{2}\delta_j{}^k & 0 \end{pmatrix}\begin{pmatrix} m_k \\ n^l \end{pmatrix} = m_k n^k \tag{2.35}$$

As Ξ naturally induces a nondegenerate metric (also denoted by Ξ) of signature (d,d) the $2d$ dimensional lattice is denoted by $\Gamma^{(d,d)}$:

$$\Xi\big(p(\vec{k},\vec{l}),\, p(\vec{n},\vec{m})\big) = \big(k_i,\, l^j\big)\begin{pmatrix} 0 & \delta^i{}_l \\ \delta_j{}^k & 0 \end{pmatrix}\begin{pmatrix} m_k \\ n^l \end{pmatrix} \tag{2.36}$$

We note that lattice $\Gamma^{(d,d)}$ is self dual w.r.t. Ξ. The scalar product Ξ is clearly $\mathbb{Z}$-valued, and furthermore the norm Ξ of a vector is an even number. A lattice with this property is called called *even*. A part of the physical relevance of Ξ is clear by the definition (2.35): In order to fulfill the physical state condition $H - \widetilde{H}\,|\text{phys}\rangle = 0$ we require:

$$H_{\text{bos, osc}} - \widetilde{H}_{\text{bos, osc}} = -\frac{\Xi\big(p(\vec{n},\vec{m})\big)}{2} \qquad \text{on physical states} \tag{2.37}$$

2.2.1 Moduli-space of toroidal compactifications, T-duality group and symmetries

For the modular invariance of the partition function $Z_{T^d}(q,\bar{q})$ the above equality needs to hold only $\mod \mathbb{Z}$. Therefore the fact that $\Gamma^{(d,d)}$ is even ensures the modular invariance of the partition function. Since any other even self-dual lattice of signature (d,d) can be reached by performing an $O(d,d,\mathbb{R})$ rotation on a given lattice $\Gamma^{(d,d)}$, the moduli space should be locally isomorphic to this Lorentz group. However separate $O(d,\mathbb{R})$ rotations (which implicitly transform both the d-dimensional lattice Γ^d, i.e. the metric G_{ij}, as well as the B-field) on the left- and the right-movers do not change the spectrum (cf. eq. (2.27) and (2.30)) and are therefore (at this level, i.e. one-loop vacuum) physically irrelevant. These $O(d,\mathbb{R})$ rotations leave not only the spectrum, but also the mass of an individual state $|m,n\rangle$ invariant. Furthermore there are rotations, which leave the whole spectrum, but not necessarily the mass of the individual states $|m,n\rangle$ invariant, thereby leading to an equivalent theory (at this level again), too. These are exactly the elements of $O(d,d,\mathbb{Z})$, the so called *T-duality group* (or Target space duality group) of the d-torus. $O(d,d,\mathbb{Z})$ transformations only permute the basis vectors of $\Gamma^{d,d}$ (possibly changing the orientation of a given, ordered basis). Therefore the moduli space of toroidal compactifications with constant G and B takes the form:

$$\mathcal{M}_{T^d} \simeq \frac{O(d,d,\mathbb{R})}{O(d,\mathbb{R}) \times O(d,\mathbb{R}) \times O(d,d,\mathbb{Z})} \tag{2.38}$$

It has been shown that the T-duality group $O(d,d,\mathbb{Z})$ has a well defined action on the oscillators α_k, $\tilde{\alpha}_l$, too (which respects the commutation relations (2.21)). We shall mention that under world sheet parity $\Omega : \sigma \mapsto -\sigma$ which has the effect:

$$\alpha_n \overset{\Omega}{\longleftrightarrow} \tilde{\alpha}_n \qquad\qquad \begin{pmatrix} m_k \\ n^l \end{pmatrix} \overset{\Omega}{\longrightarrow} \begin{pmatrix} \delta_k{}^i & -2B_{kj}/\alpha' \\ 0 & -\delta^l{}_j \end{pmatrix} \begin{pmatrix} m_i \\ n^j \end{pmatrix} \tag{2.39}$$

the scalar product Ξ changes its sign. Even though the mass formula (2.30) is invariant under the above transformation, for Ω to be a symmetry (and not just a duality), B is quantized (cf. [36]):

$$B_{ki}/\alpha' \in \frac{1}{2} \cdot \mathbb{Z} \tag{2.40}$$

such that the lattice $\Gamma^{(d,d)}$ is mapped to itself. The world sheet parity Ω should not be confused with the following kind of $SO(d,d,\mathbb{Z})$ transformation:

$$\Theta : \begin{pmatrix} m_k \\ n^l \end{pmatrix} \longrightarrow \begin{pmatrix} \delta_k{}^i & \theta_{kj} \\ 0 & \delta^l{}_j \end{pmatrix} \begin{pmatrix} m_i \\ n^j \end{pmatrix}, \quad \theta_{kj} \in \mathbb{Z}, \quad \theta = -\theta^T \tag{2.41}$$

which is equivalent to shifting $B_{ij}/\alpha' \to B_{ij}/\alpha' - \theta_{ij}$. Even though the spectrum is unchanged under Θ, it is in general not a symmetry of the theory since states $|p\rangle$ are mapped to states of different masses. If we want to mod out by the world sheet parity Ω by using this duality, we only need to distinguish the cases:

$$B_{kl} \in \{0,\, \tfrac{1}{2}\}, \qquad l, i = 1 \dots d \tag{2.42}$$

We will consider these constructions later (cf. section 2.2.2.2 and chap. 5-7).

The partition function is constructed according to eq. (2.10). The translation group $G \simeq \Gamma^d \simeq \mathbb{Z}^d$ is abelian but infinite. We have to be careful with the regularization of the projector P_{Γ^d}. V_d is the regularized d-dimensional volume (cf. (2.4)) while N is the order of Γ, which is infinite as well, but the ratio is just the volume V_Γ of the elementary d-cycle:

$$\begin{aligned} P_{\Gamma^d} &= \frac{V_d}{N} \sum_{2\pi s^j \in \Gamma^d} \exp\bigl(i2\pi s^j \Pi_j\bigr) && (2.43) \\ &= V_\Gamma \sum_{2\pi s^j \in \Gamma^d} \exp\bigl(i2\pi s^j \bigl(\sqrt{2/\alpha'} G_{jk} p^k + (1/\alpha') B_{jk} n^k\bigr)\bigr) && (2.44) \\ &= V_\Gamma \sum_{2\pi s^j \in \Gamma^d} \epsilon(s^j, n^k) \cdot \exp\bigl(i2\pi s^j \sqrt{2/\alpha'} G_{jk} p^k\bigr) && (2.45) \end{aligned}$$

$$\text{with } \epsilon(s^j, n^k) \equiv \exp\bigl(i(2\pi/\alpha') s^j B_{jk} n^k\bigr) \tag{2.46}$$

(2.45) is just a rewriting of (2.44). However $\epsilon(s^j, n^k)$ is recovered as the discrete torsion introduced at the end of section 2. Consistency condition (2.11) is fulfilled due to the defining properties of exp. Taking into account that B_{ij} is antisymmetric, (2.12) and (2.13) are obeyed. The integral $\int d^d p$ over the projector (2.43) restricts the *canonical* momenta to lie on the dual lattice $\Gamma^{d\,*}$ whereas in the torsion form (2.45) the *kinematical* momentum p is restricted to the dual torus lattice. After we perform the p integration, the partition function for the d real bosons takes the form ($\tau = \tau_1 + i\tau_2$):

$$Z_{T^d}(q,\bar{q}) = \mathrm{tr}\big(q^{H_{\mathrm{osc}}} + \bar{q}^{\widetilde{H}_{\mathrm{osc}}}\big) \cdot \sum_{\vec{n}\in\mathbb{Z}^d} \sum_{\vec{m}\in\mathbb{Z}^d} q^{H_\Gamma} + \bar{q}^{\widetilde{H}_\Gamma} \tag{2.47}$$

$$= |\eta(q)|^{-2d} \cdot \sum_{\vec{n},\,\vec{m}\in\mathbb{Z}^d} \exp\big(-2\pi\tau_2(H_\Gamma + \widetilde{H}_\Gamma)\big) \exp\big(2i\pi\tau_1(H_\Gamma - \widetilde{H}_\Gamma)\big) \tag{2.48}$$

We will prove its modular invariance. We will first consider invariance under $T : \tau \mapsto \tau + 1$. From (A.6) (p. 207) we see that the oscillator part transforms trivially (because of $|\eta|^2$ which eliminates the twelfth root of unity). The shift of τ_1 in the lattice part introduces a phase, that is however trivial as the lattice is even. To investigate the transformation of $Z_{T^d}(q,\bar{q})$ under $S : \tau \to -1/\tau$ ($\Rightarrow \tau_1 \to -\tau_1/|\tau|^2$ and $\tau_2 \to \tau_2/|\tau|^2$) we note that we can split the exponential of the lattice part as follows:

$$\frac{\tau_2}{|\tau|^2} 2(H_\Gamma + \widetilde{H}_\Gamma) = \frac{\tau_2}{|\tau|^2}\Big((\vec{m} - B\vec{n})^T \alpha' G^* (\vec{m} - B\vec{n}) + \vec{n}\frac{G}{\alpha'}\vec{n}\Big) \tag{2.49}$$

$$-i\frac{\tau_1}{|\tau|^2}(H_\Gamma - \widetilde{H}_\Gamma) = -i\frac{\tau_1}{|\tau|^2}\vec{n}\cdot(\vec{m} - B\vec{n}) \tag{2.50}$$

In (2.50) we have added a term which vanishes because of B being anti-symmetric. Therefore we can apply the Poisson-resummation formula (A.14) (p. 208) for the $\vec{m}$ resummation. For the lattice part we obtain after this resummation:

$$\left(\frac{|\tau|^2}{\tau_2}\right)^{\frac{d}{2}} \sqrt{\frac{|G|}{\alpha'}} \sum_{\vec{n}\in\mathbb{Z}^d} \sum_{\vec{w}\in\mathbb{Z}^d} e^{-i2\pi\vec{w}B\vec{n}} e^{\frac{\pi}{\tau_2}(\vec{n}+\tau_1\vec{w})^T \frac{G}{\alpha'}(\vec{n}+\tau_1\vec{w}) + \tau_2\vec{w}\frac{G}{\alpha'}\vec{w}} \tag{2.51}$$

A second Poisson resummation (now in the opposite direction) with respect to the $\vec{n}$ sum turns the above expression into:

$$|\tau|^d \sum_{\vec{v}\in\mathbb{Z}^d} \sum_{\vec{w}\in\mathbb{Z}^d} e^{i2\pi\tau_1\vec{v}\vec{w}} e^{-\pi\tau_2(\vec{v}-B\vec{w})^T\alpha' G^*(\vec{v}-B\vec{w}) + \tau_2\vec{w}\frac{G}{\alpha'}\vec{w}} \tag{2.52}$$

where we surpressed a vanishing expression $\propto i\pi\tau_1\vec{w}B\vec{w}$. Taking into account that $|\eta(q)|^{2d}$ with $q = \exp(i2\pi(-1/\tau))$ equals $|\tau|^{-d}|\eta(q)|^{2d}$ with $q = \exp(i2\pi\tau)$ (cf. (A.5), p. 206), we have proven the modular invariance of a bosonic string compactified on a torus that is described by constant background fields G and B. The fermionic part is untouched by the torus compactification. This is due to the fact that the world sheet fermions (in the RNS-formalism) are insensitive to space-time translations. The complete partition function of the super string is merely a product of the toroidal bosonic partition function times the unchanged fermionic partition function.

Before we present as a concrete example the two-torus T^2 we want to mention that it can be shown that the duality group $O(d,d,\mathbb{Z})$ of string theory compactified on T^d is also preserved by string interactions and at higher loop orders. By considering higher loop vacuum amplitudes it turns out that also the dilaton VEV is transformed under the duality group. First work on the lattice $\Gamma^{(d,d)}$ has been done by Narain, Sarmadi and Witten [37,38]. Therefore $\Gamma^{(d,d)}$ is commonly called a *Narain lattice*. The construction generalizes naturally to heterotic compactifications. There the Narain lattice $\Gamma^{(d+16,d)}$ is $2d+16$ dimensional and has the indicated signature.

2.2.2 Compactification on T^2, T-duality and symmetries

The main part of the work in this thesis consists of orientifolds which are derived either from:

$$\text{Type II on } T^2 \times T^2 (\times T^2) \tag{2.53}$$

or from orbifolds of this special torus. Thus we will have a closer look on string theory on T^2, especially the lattice part. The Narain lattice $\Gamma^{(2,2)}$ of T^2 is four dimensional. Its Lorentz group $SO(2,2,\mathbb{R}) \simeq SL(2,\mathbb{R}) \times SL(2,\mathbb{R})$ has dimension $3+3=6$. Excluding world sheet parity for the moment, which changes the sign of Ξ, we note that the T-duality group is a semi-direct product of the normal subgroup $N = SL(2,\mathbb{Z})_1 \times SL(2,\mathbb{Z})_2$ and an $H = \mathbb{Z}_2 \times \mathbb{Z}_2'$ subgroup sharing only the identity with N:

$$SO(2,2,\mathbb{Z}) = \big(SL(2,\mathbb{Z})_1 \times SL(2,\mathbb{Z})_2\big) \rtimes \big(\mathbb{Z}_2 \times \mathbb{Z}_2'\big) \tag{2.54}$$

The (real two-dimensional) fundamental representation of $SL(2,\mathbb{Z})$ is equivalent to its complex one-dimensional representation. We embed the two $SL(2,\mathbb{Z})$ in the following way: $SL(2,\mathbb{Z})_1$ acts on $\tau \hookrightarrow SO(2,2,\mathbb{R})$ by:[13]

$$\begin{array}{ccc} SL(2,\mathbb{Z})_1 \times SO(2,2,\mathbb{R}) & \longrightarrow & SO(2,2,\mathbb{R}) \\ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)(\tau) & \longmapsto & \frac{a\tau+b}{c\tau+d} \end{array} \tag{2.55}$$

with

$$\tau \equiv \tau_1 + i\tau_2 \equiv \frac{G_{12}}{G_{11}} + i\frac{\sqrt{\det G}}{G_{11}} \tag{2.56}$$

The action of the second $SL(2,\mathbb{Z})$ is defined analogously by the following embedding of the modular parameter $\rho \hookrightarrow SO(2,2,\mathbb{R})$:

$$\rho \equiv \rho_1 + i\rho_2 \equiv B_{12}/\alpha' + i\sqrt{\det(G/\alpha')} \tag{2.57}$$

The two $\mathbb{Z}_2$ subgroups are:

$$\mathbb{Z}_2 = \left\{ \mathrm{Id}, D | \tau \overset{D}{\longleftrightarrow} \rho \right\} \tag{2.58}$$

$$\mathbb{Z}_2' = \left\{ \mathrm{Id}, R | (\tau,\rho) \overset{R}{\longleftrightarrow} -(\bar{\tau},\bar{\rho}) \right\} \tag{2.59}$$

[13] τ parametrizes $SL(2,\mathbb{R})/SO(2,\mathbb{R})$ and can therefore be embedded into $SO(2,2,\mathbb{R})$.

In order to discover the role of the subgroups in terms of their action on the background fields we will rewrite the transformations. The $SL(2,\mathbb{Z})_1$ is purely geometric. It just describes the change to a new basis of the torus lattice Γ^2. We define the mass matrix M^2 by

$$\frac{M^2}{2} \equiv \frac{1}{2}\begin{pmatrix} \alpha' G^{ik} & -B^i{}_l \\ B_j{}^k & \frac{1}{\alpha'}\left(G - B^2\right)_{jl} \end{pmatrix} \tag{2.60}$$

Then $SL(2,\mathbb{Z})_1$ transforms M^2 under (2.55) in the following way (not distinguishing lower and upper indices in M^2 this time):

$$M^2 \mapsto S^T M^2 S, \qquad S = \left(\begin{array}{c|c} \begin{smallmatrix} d & -c \\ -b & a \end{smallmatrix} & 0 \\ \hline 0 & \begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \end{array}\right), \qquad \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL(2,\mathbb{Z}) \tag{2.61}$$

where we have taken into account that a linear map transforms the dual basis with the transposed inverse. An alternative expression of the mass^2 matrix (2.60) in terms of the parameters τ (called *complex structure*[14]) and ρ (commonly denoted as *Kähler structure*[14]) is (c.f. eq. (2.31)):

$$\frac{M^2}{2} = \left(p_+^2 + p_-^2\right) \tag{2.62}$$

$$\begin{aligned} p_+^2 &= \frac{1}{4}\frac{1}{\tau_2\rho_2}\left|(\tau m_1 - m_2) - \bar{\rho}(n_1 + \tau n_2)\right|^2 \\ p_-^2 &= \frac{1}{4}\frac{1}{\tau_2\rho_2}\left|(\tau m_1 - m_2) - \rho(n_1 + \tau n_2)\right|^2 \end{aligned} \qquad n_i, m_i \in \mathbb{Z} \tag{2.63}$$

The $SL(2,\mathbb{Z})_2$ is stringy. The generators have the following correspondences:

$$\rho \to \rho + 1 \qquad\qquad \rho \to -\frac{1}{\rho} \tag{2.64}$$

$$B_{12}/\alpha' \to B_{12}/\alpha' + 1 \qquad\qquad B_{12}/\alpha' \to -\frac{\alpha' B_{12}}{B_{12}^2 + \det G} \tag{2.65}$$

$$\sqrt{\det(G/\alpha')} \to \frac{\alpha'\sqrt{\det G}}{B_{12}^2 + \det G}$$

$$S = \left(\begin{array}{c|c} \mathbb{1}_2 & \begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix} \\ \hline 0 & \mathbb{1}_2 \end{array}\right) \qquad\qquad S = \left(\begin{array}{c|c} 0 & \mathbb{1}_2 \\ \hline \mathbb{1}_2 & 0 \end{array}\right) \tag{2.66}$$

However, the $\det g = 1$-condition ($g \in SL(\mathbb{Z})$) is not so easily imposed, so that we have presented only the action of the generators of the latter $SL(2,\mathbb{Z})_2$ transformation on the background fields and not the action of a general element of $SL(2,\mathbb{Z})_2$. $\rho \to \rho+1$ represents an integer shift on B. $\rho \to -1/\rho$ corresponds to T-duality along both directions of the torus. The two nontrivial elements D

[14]The flat torus is a *Calabi-Yau* space. τ is called the complex structure as it parametrizes the different complex structures of the two-torus (cf. fig. 2.1). The origin of the name *Kähler structure* comes from the observation that ρ can be interpreted as a complex $(1,1)$-form. In CY-spaces $(1,1)$ forms correspond to deformations of the *Kähler structure* (that preserve the CY-property). This point is a bit subtle as the $(1,1)$ forms that describe the CY-preserving deformations of the metric are real. However they can be complexified by combining them with the NSNS B-field (cf. [39]).

and R of the $\mathbb{Z}_2 \times \mathbb{Z}_2'$ subgroup are a stringy T-duality respectively a reflection in either the e_1 or e_2 direction:[15]

$$
\begin{array}{cc}
D & R \\
\dfrac{G_{12}}{G_{11}} \leftrightarrow B_{12}/\alpha' & G_{12} \leftrightarrow -G_{12} \\
\dfrac{\sqrt{\det G}}{G_{11}} \leftrightarrow \sqrt{(\det G)/\alpha'} & B_{12} \leftrightarrow -B_{12} \\
S_D = \left(\begin{array}{cc|cc} 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ \hline -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right) & S_R = \left(\begin{array}{cc|cc} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \hline 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right)
\end{array}
\tag{2.67}
$$

2.2.2.1 Points of enhanced symmetry in the moduli space $SO(2, 2, \mathbb{R})$

As toroidal orbifolds are obtained by modding out symmetries of the torus, we are especially interested in points of the moduli space which are fixed by certain elements of the T-duality group. In these cases the dualities are enhanced to symmetries of the string compactification, so that they can be modded out. In chapter 7 we consider a Z_4 orbifold which is obtained from a special T^6:

$$
\frac{T^2 \times T^2 \times T^2}{\mathbb{Z}_4} \tag{2.68}
$$

The $\mathbb{Z}_4$ acts on each of the first two T^2s as $\exp(i2\pi/4)$ and as $\exp(-i2\pi/2)$ on the last torus (written in terms of complexified coordinates). The $\exp(i2\pi/4)$ rotation restricts the moduli in the following way:

$$
\mathbb{Z}_4 : \qquad \tau = i \qquad \Longrightarrow G_{11} = G_{22}, \qquad G_{12} = 0 \tag{2.69}
$$

The corresponding matrix acting on M^2 (leaving it invariant) and consequently on the vectors of $\Gamma^{(d,d)}_{\mathbb{Z}_4}$ is:

$$
S_{\mathbb{Z}_4} = \left(\begin{array}{cc|cc} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{array}\right) \tag{2.70}
$$

This means that the torus lattice has a basis consisting of two vectors of equal but unrestricted length, making an angle of $\pi/2$. The $\mathbb{Z}_2$ symmetry $\exp(-i2\pi/2)$ does not restrict the values of G and B. It multiplies all vectors in the Narain lattice by -1: $S_{\mathbb{Z}_2} = -\mathbb{1}_4$. In section 2.4 we will consider an orbifold where we divide out a special four torus $T^2 \times T^2$ by the product $\mathbb{Z}_3^L \times \mathbb{Z}_3^R$ with one $\mathbb{Z}_3$ only acting on the left-moving degrees of freedom and the other only on the right-moving part of the string. Since this direct product contains the symmetric $\mathbb{Z}_3$ as a subgroup with the generator acting as $\exp(i2\pi/3)$, we will first look at the lattice having this symmetry:

$$
\mathbb{Z}_3 : \qquad \tau = \frac{1}{2} + i\frac{\sqrt{3}}{2} \qquad \Longrightarrow G_{11} = G_{22}, \qquad G_{12} = \frac{1}{2} G_{11} \tag{2.71}
$$

[15] S_D represents usual T-duality along e_1 times a reflection in the same direction, while S_R is a reflection along e_1

This describes a lattice admitting a basis with vectors of equal length and a mutual angle of $2\pi/3$. It is up to a scale factor the root lattice of the $SU(3)$ Lie algebra. The action of the symmetric $\mathbb{Z}_3$ is given by (in terms of the basis which was described above):

$$S_\theta = S_{\mathbb{Z}_3} = \left(\begin{array}{c|c} \begin{smallmatrix} 0 & -1 \\ 1 & -1 \end{smallmatrix} & 0 \\ \hline 0 & \begin{smallmatrix} -1 & -1 \\ 1 & 0 \end{smallmatrix} \end{array}\right) \tag{2.72}$$

We note that the two-torus described by (2.72) admits in addition a $\mathbb{Z}_6$ symmetry, namely the geometric rotation by $\pi/3$. Since the asymmetric $\mathbb{Z}_3^L \times \mathbb{Z}_3^R$ can be generated by the symmetric $\exp(i2\pi/3)$ and an element $\hat{\theta}$ rotating the left-movers by $\exp(i2\pi/3)$ and the right-movers by the reversed angle, we will search for Narain lattices $\Gamma^{(d,d)}$ admitting this latter symmetry. Especially the associated matrix $S_{\hat{\theta}}$ acting on the (m_i, n^j) has to have integer entries. In principle we could determine the form of $S_{\hat{\theta}}$ by mapping (m_i, n^j) to (p_+, p_-) (where the form of the asymmetric $\hat{\theta}$ is explicitly known), performing the rotation $\hat{\theta}$ and mapping back to the (m_i, n^j) basis. The map between the two basis is described by Υ and Υ^{-1} (eq. (2.33) and (2.34)):

$$S_\theta(p_+, p_-) = \Upsilon^{-1} S_\theta(m, n) \Upsilon \tag{2.73}$$

However, we will proceed differently. We know (from (2.28), (2.67)) that at the self dual radius with vanishing B-field, the T-Duality D from (2.67) reflects the X_2 coordinate on the right-movers, leaving the rest unchanged. By acting with $D^{-1}\theta D$ we achieve (on the T^2 at the self dual radius, $B = 0$) that the right-movers get rotated in the inverse direction w.r.t. the left-movers (if we choose the metric of the dual T^2 to admit the symmetric $\mathbb{Z}_3$ action). Via the D-duality (2.58), (2.67) the metric (2.71) of the symmetric $\mathbb{Z}_3$ maps to the background fields of the asymmetric $\widehat{\mathbb{Z}}_3$ (in terms of the old background fields):

$$\widehat{\mathbb{Z}}_3: \qquad \frac{G_{ij}^{\widehat{\mathbb{Z}}_3}}{\alpha'} = \begin{pmatrix} \frac{\alpha'}{G_{11}} & \frac{B_{12}}{G_{11}} \\ \frac{B_{12}}{G_{11}} & \alpha'\left(\frac{3}{4}G_{11} + \frac{(B_{12})^2}{G_{11}}\right) \end{pmatrix} \qquad \frac{B_{12}^{\widehat{\mathbb{Z}}_3}}{\alpha'} = 1/2 \tag{2.74}$$

$$\rho = \frac{1}{2} + i\frac{\sqrt{3}}{2} \tag{2.75}$$

The $\widehat{\mathbb{Z}}_3$ action takes the following form in the momentum and winding number basis:

$$S_{\hat{\theta}} = S_D^{-1} S_\theta S_D = \left(\begin{array}{c|c} -\mathbb{1} & \begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix} \\ \hline \begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix} & 0 \end{array}\right) \tag{2.76}$$

The $\widehat{\mathbb{Z}}_3$ action on the left- and right-moving momenta $(e^k(p_+)_k, (e^l p_-)_l$ is:

$$S_{\hat{\theta}}(p_+, p_-) = \Upsilon_{\widehat{\mathbb{Z}}_3} S_\theta \Upsilon_{\widehat{\mathbb{Z}}_3}^{-1} = \left(\begin{array}{c|c} \theta & 0 \\ \hline 0 & \theta^{-1} \end{array}\right), \quad \theta_i{}^j = \begin{pmatrix} -\left(\frac{B_{12}}{G_{11}} + \frac{1}{2}\right) & \frac{\alpha'}{G_{11}} \\ \frac{4B_{12}^2 + 3G_{11}^2}{4\alpha' G_{11}} & \frac{B_{12}}{G_{11}} - \frac{1}{2} \end{pmatrix} \tag{2.77}$$

Therefore the action $S_{\hat{\theta}}$ represents for all allowed backgrounds (even those with $B \neq 0$ in the D dual geometry) an asymmetric rotation of the form $(\theta_L, \theta_R) =$

(θ, θ^{-1}). If we want to have the full $\widehat{\mathbb{Z}}_3^L \times \widehat{\mathbb{Z}}_3^R$ symmetry, we are restricted to backgrounds of the form:

$$\mathbb{Z}_3^L \times \mathbb{Z}_3^R : \qquad \tau = \rho = \frac{1}{2} + i\frac{\sqrt{3}}{2} \tag{2.78}$$

In the same way we get for the asymmetric $\widehat{\mathbb{Z}}_4$:

$$\widehat{\mathbb{Z}}_4 : \qquad \rho = i \tag{2.79}$$

If we want to maintain the asymmetric $\widehat{\mathbb{Z}}_4$ as well as symmetric $\mathbb{Z}_4$ action we need:

$$\mathbb{Z}_4 \times \widehat{\mathbb{Z}}_4 : \qquad \tau = \rho = i \tag{2.80}$$

However $\mathbb{Z}_4 \times \widehat{\mathbb{Z}}_4$ does not generate the full $\mathbb{Z}_4^L \times \mathbb{Z}_4^R$ since it does not contain elements like $(\theta_L, \theta_R) = (\theta, \mathrm{Id})$. In chapter 5 we will investigate the orientifold of the asymmetric $\mathbb{Z}_3^L \times \mathbb{Z}_3^R$ orbifold.[16]

2.2.2.2 The world-sheet-parity on T^2

We mentioned already that the world-sheet parity $\Omega : \sigma \to -\sigma$ is a symmetry of the compactified theory, iff the B-field obeys condition (2.40). This means especially that Ω is a symmetry of the $\mathbb{Z}_3^L \times \mathbb{Z}_3^R$-symmetric background (2.78). In the next chapter we will see how symmetries involving Ω are gauged, leading to so called *orientifolds*. One can also combine Ω with an element s which acts on the space(-time) and more generally, on the Narain lattice, s.th. the resulting $s\Omega$ is still a symmetry of the theory. In chapter 6 and 7 we will gauge by $\bar{\sigma}\Omega$ with $\bar{\sigma}$ acting as:

$$\begin{pmatrix} X_1 \\ X_2 \end{pmatrix} \xrightarrow{\bar{\sigma}} \begin{pmatrix} X_1 \\ -X_2 \end{pmatrix} \qquad \text{or in complex coords.:} \quad Z \xrightarrow{\bar{\sigma}} \bar{Z} \tag{2.81}$$

This action is a symmetry of bosonic string theory on T^2 iff the complex structure τ fulfills either

$$\tau_1 = 0 \quad \text{or} \quad \tau_1 = 1/2 \tag{2.82}$$

For $\bar{\sigma}\Omega$ to be a symmetry of the superstring, it has to be compatible with the GSO projection: $\bar{\sigma}$ acts on a Ramond zero mode $|\, s\,\rangle$ by $s = \pm 1/2 \xrightarrow{\bar{\sigma}} s = \mp 1/2$. If the GSO-projection takes the form:[17]

$$|\, s_0 \ldots s_n\,\rangle_{\mathrm{L}} : \qquad \sum_{i=0}^{n} \varepsilon_i s_i^{\mathrm{L}} = a \mod 2, \qquad \varepsilon_i \in \{-1, 1\} \tag{2.83}$$

[16] The *four* dimensional $T^6/(\mathbb{Z}_4 \times \widehat{\mathbb{Z}}_4)$ orientifold is presumably fraught with the same problems as the four dimensional $T^6/\mathbb{Z}_4$ Ω-orientifold of [40]. We will make some comments about this in the following chapter.

[17] This is true for sectors which are untwisted or twisted by a left-right symmetric twist. a is an integer which can be chosen to be one.

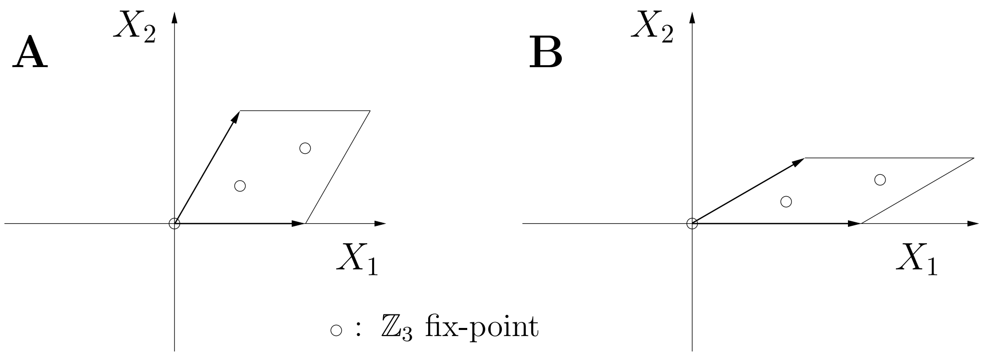


Figure 2.3: Left: **A** torus with complex structure $\tau = \frac{1}{2} + i\frac{\sqrt{3}}{2}$. Right: **B** torus with $\tau = \frac{1}{2} + i\frac{1}{2\sqrt{3}}$. The $\mathbb{Z}_3$ fix points are depicted as well.

on left-movers, it is for right-movers:

$$|\, s_0 \dots s_n \rangle_{\mathrm{R}} : \qquad \sum_{i=0}^{n} \varepsilon_i \bar{\sigma}\big(s_i^{\mathrm{R}}\big) = a \mod 2, \qquad \varepsilon_i \in \{-1, 1\} \tag{2.84}$$

$\bar{\sigma}$ multiplies the s_i with -1 on the complex planes on which it acts by (2.81). We can absorb this action in a redefinition of the ε_i. For $\bar{\sigma}\Omega$ orientifolds with even number of planes with $\bar{\sigma}$ action (2.81), left- and right-movers have the same GSO-projection. Therefore in this case, $\bar{\sigma}\Omega$ is a symmetry of Type IIB theory. The converse is true for an odd number of complex planes on which $\bar{\sigma}$ acts by complex conjugation. In the latter case $\bar{\sigma}\Omega$ is a symmetry of Type IIA theory. We will make one comment on the $\mathbb{Z}_3$-symmetric torus (2.71). There is an (equivalent) torus obtained from (2.71) by transforming the background by the element

$$TST : \tau \to \tfrac{\tau}{\tau+1} \tag{2.85}$$

of the T-duality group $SL(2, \mathbb{Z})_1$. It leads to:

$$\tau_B = \frac{1}{2} + i\frac{1}{2\sqrt{3}} \qquad \Longrightarrow G_{11} = 3G_{22}, \qquad G_{12} = \frac{1}{2}G_{11} \tag{2.86}$$

while leaving ρ unchanged. Even though the orbifold theory is completely equivalent, the gauging of $\bar{\sigma}\Omega$ leads to inequivalent models. We call the torus obtained from (2.85) the **B** torus and the "usual" $\mathbb{Z}_3$-torus (2.71) the **A** torus. Both $\mathbb{Z}_3$ symmetric tori are depicted in figure 2.3. The asymmetric $\mathbb{Z}_3$ rotation $\hat{\theta}$ is now again obtained by D-duality (2.58). It leads to the background:

$$\begin{aligned}
\widehat{\mathbb{Z}}_3 : \qquad \frac{G_{ij}^{\widehat{\mathbb{Z}}_3}}{\alpha'} &= \begin{pmatrix} \frac{\alpha'}{G_{11}} & \frac{B_{12}}{G_{11}} \\ \frac{B_{12}}{G_{11}} & \alpha'\Big(\frac{1}{12}G_{11} + \frac{(B_{12})^2}{G_{11}}\Big) \end{pmatrix} \qquad \frac{B_{12}^{\widehat{\mathbb{Z}}_3}}{\alpha'} = 1/2 \\
\tau &= \frac{B_{12}}{G_{11}} + i\frac{G_{11}}{2\sqrt{3}} \qquad\qquad \rho = \frac{1}{2} + i\frac{1}{2\sqrt{3}}
\end{aligned} \tag{2.87}$$

Symmetry under the asymmetric $\mathbb{Z}_3^L \times \mathbb{Z}_3^R$ rotation group then requires the background:

$$\mathbb{Z}_3^L \times \mathbb{Z}_3^R : \qquad \tau_B = \rho_B = \frac{1}{2} + i\frac{1}{2\sqrt{3}} \tag{2.88}$$

Instead of acting with $\bar{\sigma}\Omega$ (or Ω) on the **B** torus, we can alternatively implement the action of (2.85) directly into $\bar{\sigma}\Omega$ (or Ω):

$$\bar{\sigma}\Omega_B \equiv (TST)^{-1}\,\bar{\sigma}\Omega\,(TST) \tag{2.89}$$

The action of $\bar{\sigma}\Omega$ takes the following form on the (m, n)-basis of the $\mathbb{Z}_3$ background (2.71):

$$\mathbb{Z}_3: \qquad \bar{\sigma}\Omega_{\mathbf{A}} = \left(\begin{array}{c|c} \begin{smallmatrix} -1 & 0 \\ -1 & 1 \end{smallmatrix} & 0 \\ \hline 0 & \begin{smallmatrix} 1 & 1 \\ 0 & -1 \end{smallmatrix} \end{array}\right) \qquad \bar{\sigma}\Omega_{\mathbf{B}} = \left(\begin{array}{c|c} \begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix} & 0 \\ \hline 0 & \begin{smallmatrix} 0 & -1 \\ -1 & 0 \end{smallmatrix} \end{array}\right) \tag{2.90}$$

Under the T-duality D (2.58) these symmetries become symmetries of the $\widehat{\mathbb{Z}}_3$ background (2.74):[18]

$$\widehat{\mathbb{Z}}_3: \qquad \Omega_{\mathbf{A}} = \left(\begin{array}{c|c} \mathbb{1} & \begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix} \\ \hline 0 & -\mathbb{1} \end{array}\right) \qquad \Omega_{\mathbf{B}} = \left(\begin{array}{c|c} 0 & \begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix} \\ \hline \begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix} & 0 \end{array}\right) \tag{2.91}$$

At the $\mathbb{Z}_3^L \times \mathbb{Z}_3^R$ symmetric point (2.78) all four actions of $(\bar{\sigma})\Omega$ (2.90),(2.91) are symmetries of the theory and can be gauged. These symmetries extend to the sector of the zero- and oscillator-modes as well. While the bosonic zero-modes (i.e. the center of mass coordinates) are sensitive to the distinction between **A** and **B** lattices (equivalently: sensitive to $..\Omega_{\mathbf{A}}$ and $..\Omega_{\mathbf{B}}$) as well, fermionic and oscillator parts are unaffected in the end.[19]

The $\bar{\sigma}\Omega$ action has the interesting property that it maps a g-twisted sector onto itself (g a geometric rotation in the complex plane, on that $\bar{\sigma}$ acts by complex conjugation). This is in contrast to the pure world sheet parity Ω that maps a g-twisted sector to the g^{-1} twisted sector (which is different if $g \notin \mathbb{Z}_2$). Another interesting feature of the $\bar{\sigma}\Omega$ action is, that its invariant closed-string oscillator excitations on each T^2 are of the form:[20]

$$\left|\alpha_{i_1}^{k_1}\alpha_{i_2}^{k_2}\ldots\alpha_{i_n}^{k_n}, \bar{\alpha}_{j_1}^{l_1}\bar{\alpha}_{j_2}^{l_2}\ldots\bar{\alpha}_{j_m}^{l_n}\right\rangle \otimes \left|\bar{\tilde{\alpha}}_{i_1}^{k_1}\bar{\tilde{\alpha}}_{i_2}^{k_2}\ldots\bar{\tilde{\alpha}}_{i_n}^{k_n}, \tilde{\alpha}_{j_1}^{l_1}\tilde{\alpha}_{j_2}^{l_2}\ldots\tilde{\alpha}_{j_m}^{l_n}\right\rangle \tag{2.92}$$

We used here a complexified basis for the oscillators. In this basis a (symmetric) rotation $g \in U(1) \simeq SO(2)$ acts as

$$g: \; Z \mapsto e^{i2\pi\phi_g} Z \qquad \bar{Z} \mapsto e^{-i2\pi\phi_g}\bar{Z} \tag{2.93}$$

[18] $\Omega_{\mathbf{A}}$ is indeed a symmetry of the $\mathbb{Z}_3$ background (2.71) if we impose in addition: $B_{12}/\alpha' = 1/2$ (cf. (2.40)). $\Omega_{\mathbf{B}}$ requires however the full $\mathbb{Z}_3^L \times \mathbb{Z}_3^R$ symmetric point (2.78) in order to be a symmetry.

[19] Of course the coordinate representation is different even for the oscillator modes. However in physical quantities like partition functions the oscillator and w.s.-fermionic parts are unaffected by the distinction between **A** and **B** lattices (or $..\Omega$-action).

[20] We present only bosonic degrees of freedom schematically, but the world sheet fermions are treated analogously, implementing the fact that fermionic occupation numbers for individual modes are either one or zero.

on the complex coordinates (2.81). This implies that $\bar{\sigma}\Omega$-invariant oscillator states (2.92) and its fermionic counterparts are automatically invariant under geometric $U(1)$ rotations g. Remark: this is in general not true for linear excitations (i.e. excitations in the Narain lattice). These properties extend naturally to direct products of two-tori.

2.3 Toroidal orbifolds

In this section we will consider orbifolds of the type:

$$\mathcal{X}_d = T^d/G \tag{2.94}$$

with G a symmetry of the torus T^d. For the sake of simplicity, we will restrict to the case of *abelian* groups G. However, we will also consider the case where G acts differently on left- and right-movers, such that the action of G is well defined on the Narain lattice $\Gamma^{(d,d)}$ and on the left- and right-moving parts. The group G can consist of rotations $\in O(d,d,\mathbb{Z})$ and translations. We will restrict in this work to the case where G is purely rotational, even though translations (or shifts) can give rise to interesting effects. The untwisted sector of the orbifold is obtained by projection on G-invariant states. The untwisted partition function consists of all trace insertions of elements $g \in G$. We will have a closer look at the sector twisted by gs, ($g \in G$, $s \in \Gamma$) in the σ direction (c.f. eq. (2.3)). These boundary conditions are solved by the following mode expansion (lattice momenta and center of mass coordinate are left out):

$$\begin{aligned} X_+^{\mathrm{osc}}(\tau+\sigma)^\mu &= i\sqrt{\frac{\alpha'}{2}}\sum_j \sum_{n_j\in\mathbb{Z}+\beta_j}{}' C_j^\mu \frac{\tilde{\alpha}^j_{-n_j}}{n_j} e^{in_j(\tau+\sigma)} \\ X_-^{\mathrm{osc}}(\tau-\sigma)^\mu &= i\sqrt{\frac{\alpha'}{2}}\sum_k \sum_{n_k\in\mathbb{Z}+\gamma_k}{}' D_k^\mu \frac{\alpha^k_{-n_k}}{n_k} e^{-in_k(\tau-\sigma)} \end{aligned} \tag{2.95}$$

We split g into (g_+, g_-), a part acting on X_+ which is g_+ and g_- acting on X_-. Then the C_j^μ are defined as Eigenvectors of g_+ with Eigenvalue $\kappa_j = \exp i2\pi\beta_j$. (Analogously, the D_j^μ are Eigenvectors of g_- with Eigenvalue $\lambda_j = \exp i2\pi\gamma_j$). Since g_+ is a rotation, the C_j form an orthogonal system (which can be normalized) w.r.t. the hermitian form that is induced by the euclidian scalar product on $\mathbb{R}^d$. They can be interpreted as Vielbeins. This is analogous to the discussion of open strings in constant background fields (cf. chapter 4). For each complex C_j there exists a complex conjugate C_{-j} with c.c. Eigenvalue $\lambda_{-j} = \bar{\lambda}_j$. The D_k fulfill analogous properties w.r.t. the rotation g_-. The oscillators obey the following commutation relations:

$$\left[\alpha^i_{l_i}, \alpha^k_{m_k}\right] = l_i \cdot \delta_{(l_i+m_k),0}\left\langle C^i, C^k\right\rangle, \quad \left[\tilde{\alpha}^i_{l_i}, \tilde{\alpha}^k_{m_k}\right] = l_i \cdot \delta_{(l_i+m_k),0}\left\langle D^i, D^k\right\rangle \tag{2.96}$$

In this case $\langle C^i, C^k\rangle$ is the inverse of $\langle C_i, C_k\rangle$. The lattice part obeys:

$$\begin{aligned} &X_+^\Gamma(\tau,\sigma) = \sqrt{\alpha'}\big(p_+(\tau+\sigma) + p_-(\tau-\sigma)\big) \\ &\text{with} \quad p_+ = g_+p_+ \quad \text{and} \quad p_- = g_-p_-, \quad \big(p_+, p_-\big) \in \Gamma^{(d,d)} \end{aligned} \tag{2.97}$$

The center of mass $x_{\rm com}$ has *a priori* no well defined splitting into left- and right-movers. Therefore g has a clear interpretation in terms of a geometric action only in the symmetric case: $(g_L, g_R) = (g, g)$. Then the boundary condition reads:

$$(\mathrm{Id} - g)x_{\rm com} = -2\pi\sqrt{\alpha'}(p_+ - p_-) + s, \quad s \in \Gamma^d \tag{2.98}$$

If $I_g \subset \Gamma^d$ is the lattice invariant under g and $N_g = \{u \in \Gamma | u \cdot v = 0 \, \forall v \in I_g\}$ the lattice perpendicular to I_g, the right hand side of (2.98) is also contained in N_g. (Proof: multiply both sides by $v \in I_g$. The left side vanishes since $v \cdot (\mathrm{Id} - g)x = (\mathrm{Id} - g^{-1})v \cdot x$.) For $g = \mathrm{Id}$ we get exactly the result of the toroidal compactification. For $g \neq \mathrm{Id}$ the com. coordinate is restricted to be fixed up to addition of a lattice vector. $x_{\rm com} + w$, with w an arbitrary lattice vector, fulfills eq. (2.98) if s is shifted by $(1-g)w$ which is a vector lying in the sub-lattice N_g. Therefore such a shifted com. coordinate is equivalent to the non shifted one. In [41] the number of inequivalent fixed points (more general: fixed planes) was thereby determined to be:[21]

$$n_{\mathrm{fix},g} = \left| \frac{N}{(1-g)\Gamma^d} \right| \tag{2.99}$$

This formula has a simple generalization for left-right asymmetric twists $g = (g_+, g_-)$: In [41] the authors gave a definition that only refers to the Narain lattice:[22]

$$n_{\mathrm{fix},g} = \sqrt{\left| \frac{N}{(1-g)\Gamma^{d,d}} \right|} \tag{2.100}$$

N from our former definition has to be replaced by the lattice which is orthogonal to the invariant sub-lattice I of the Narain lattice $\Gamma^{d,d}$. In the same publication the authors proved that (2.100) is always an integer. The proof is rather lengthy. It involves the embedding of the Narain lattice $\Gamma^{d,d}$ into a lattice of doubled dimension. Especially in the path-integral formalism the splitting into holomorphic and anti-holomorphic fields is *a priori* not possible. The number of h-invariant fixed-points in the g-twisted sector is of big importance, too. It appears as an overall constant in the sector $h\underset{g}{\square}$. In a subsequent publication (cf. [42]) the same authors determined this number to be the *square root* of:

$$C_{h,g} = \left| \frac{N}{(1-g)N^* \cup (1-h)N} \right| \tag{2.101}$$

N^* is the lattice dual to N. In the example which we present in the following, most of the numbers $C_{h,g}$ are determined by modular transformations of untwisted parts in the partition function. (By applying (2.101) in combination with modular transformations we will relate in fact all these numbers to partition functions in the (σ-) untwisted sector of the $T^4/\left(\mathbb{Z}_3^{\mathrm{L}} \times \mathbb{Z}_3^{\mathrm{R}}\right)$ orbifold.)

For the superstring there exists an analog of the mode expansion (2.97) for the twisted world-sheet fermions (in the NSR formalism). World sheet

[21] By $(1-g)\Gamma$ we mean the lattice $\{(\vec{r} - g\vec{r}) \,|\, \vec{r} \in \Gamma\}$.

[22] In fact they consider heterotic compactifications where the Narain lattice is of the more general form $\Gamma^{d,p}$.

fermions do not have extra com. degrees of freedom. As the Hilbert space is a tensor product of the fermionic and the bosonic sector, the bosonic zero-modes (i.e. the com. coordinates) determine the multiplicity of fermionic states as well. If the space-time fermionic sector (e.g. the R-sector of the heterotic string or the Ω-symmetrized NSR-sector of an orientifold) has only one massless excitation, the number of fixed points (=inequivalent com. coordinates) in this sector determines the number of chiral fermions. This is similar to the case of intersecting D-branes, where the number of chiral fermions in the respective open string sector is determined by the intersection number of the two D-branes to which the string is attached (c.f. chapter 6, 7, and [43]). There are additional conditions for a group G to be a symmetry of the superstring: G must be a symmetry not only of the bosonic and fermionic Hamiltonian (or: world-sheet energy-momentum tensor) (i.e. T_B), but also of the world sheet supercurrent (T_F). Of course G must preserve interactions, especially the OPE, as well. In addition, the partition function has to be modular invariant.

Before we will turn to an (asymmetric) example, we will summarize some well known facts about space-time supersymmetric, geometric toroidal orbifolds.

2.3.1 Space-time supersymmetric (toroidal) orbifolds

In [32, 33] orbifolds were introduced in the context of superstring theory. However this was not the first time orbifolds appeared in physics (cf. references in [32]). In mathematics they go back to Satake [44]. Our intention is to re-state (sufficient) conditions for the orbifold to be supersymmetric. Even though an orbifold is not a manifold, certain orbifolds can be deformed into a manifold. An interesting class of obifolds are those which can be deformed into a Calabi-Yau manifold, as superstrings compactified on such an orbifold yield space-time supersymmetry. A complex n-dimensional compact manifold $\mathcal{M}$ that is Calabi-Yau (i.e. Kähler and first Chern class $c_1 = 0$) admits a unique Ricci-flat metric for a given Kähler class and complex structure.[23] This does not mean that every metric on $\mathcal{M}$ is Ricci flat, but it means that such a metric exists. The property of a complex n-dimensional manifold to be kähler restricts its holonomy to be at most $U(n)$. Ricci flatness implies that the holonomy group with respect to the Ricci-flat metric is contained even in $SU(n)$, iff $c_1(\mathcal{M}) = 0$.[24] As a consequence the manifold as a spin-manifold admits one Killing-spinor (=covariantly constant spinor) of each chirality. Unbroken supersymmetry requires the supersymmetry variation of the gravitino to vanish. In the absence of an NSNS 3-form field strength H this is equivalent to the statement that the covariant derivative of the supersymmetry parameter η vanishes: $D\eta = 0$ (i.e. η is a Killing spinor). If one covariant constant spinor exists on the compactification

[23] Sometimes CY manifolds are defined as the triple $(\mathcal{M}, J, g)$ with $\mathcal{M}$ a complex n-dimensional Kähler manifold with $c_1 = 0$, J its complex structure, and g the Kähler metric, which in addition should be Ricci-flat. In section 6.5.1 we will give a third definition making explicit use of a holomorphic $(n, 0)$-form Ω which always exist on a CYn-fold.

[24] In Kähler manifolds, $SU(n)$ holonomy implies also Ricci-flatness. Conversely Ricci-flat Kähler manifolds admit $SU(n)$ holonomy, if they are simply-connected.

$\mathbb{Z}_2 : \vec{v} = (1, -1)/2$	$\mathbb{Z}_4 : \vec{v} = (1, -1)/4$
$\mathbb{Z}_3 : \vec{v} = (1, -1)/3$	$\mathbb{Z}_6 : \vec{v} = (1, -1)/6$

Table 2.1: $\mathbb{Z}_N$ groups preserving $\mathcal{N} = 1$ supersymmetry in $D = 6$.

$\mathbb{Z}_3 : \vec{v} = (1, 1, -2)/3$	$\mathbb{Z}'_6 : \vec{v} = (1, 2, -3)/6$	$\mathbb{Z}'_8 : \vec{v} = (1, 2, -3)/8$
$\mathbb{Z}_4 : \vec{v} = (1, 1, -2)/4$	$\mathbb{Z}_7 : \vec{v} = (1, 2, -3)/7$	$\mathbb{Z}_{12} : \vec{v} = (1, 4, -5)/12$
$\mathbb{Z}_6 : \vec{v} = (1, 1, -2)/6$	$\mathbb{Z}_8 : \vec{v} = (1, 3, -4)/8$	$\mathbb{Z}'_{12} : \vec{v} = (1, 5, -6)/12$

Table 2.2: $\mathbb{Z}_N$ groups preserving $\mathcal{N} = 1$ supersymmetry in $D = 4$.

space $\mathcal{M}$, it can serve as the supersymmetry parameter η. Supersymmetry requires in addition that the variation of the gluino vanishes for each gauge group. We will not pursue this second question. We note however that in the absence of non-trivial field strength H, the low energy supersymmetry-conditions require the compactification manifold $\mathcal{M}$ to be of CY-type to first order in α'. (α'-corrections from string theory deform the CY-condition only continiously, s.th. it is justified to neglect them in a first approximation.)

Like a smooth manifold, an orbifold admits a holonomy group. The holonomy group of an orbifold is closely related to the orbifold group. The holonomy group of a general real four dimensional manifold is $SO(4) \simeq SU(2) \times SU(2)$. In general, such a manifold will admit no global Killing spinors. If however the holonomy is contained in an $SU(2)$ subgroup, a single Killing spinor exists (for each chirality) and one supersymmetry will survive. Orbifold groups that are discrete subgroups of $SU(2)$, and which admit geometric action on four-dimensional tori, have been listed in [32, 33]. We list them in table 2.1. If a ten dimensional string-theory with $\mathcal{N} = 1$ supersymmetry in ten space dimensions is compactified on such real four dimensional orbifold space, it will lead to $\mathcal{N} = 1$ in six dimensions. This is the case for heterotic string and for Type I. Table 2.1 should be understood as follows: The Eigenvalues of the rotations $\theta \simeq g \in G$ are $\exp(\pm 2i\pi v_1)$ and $\exp(\pm 2i\pi v_2)$. A general orbifold-rotation $\theta \in G$ can then be described in a suitable (complexified) basis by:

$$\theta : Z_i \mapsto \exp(2\pi i v_i)\, Z_i \qquad \bar{Z}_i \mapsto \exp(-2\pi i v_i)\, \bar{Z}_i \tag{2.102}$$

Four dimensional theories with $\mathcal{N} = 1$ in four dimensions are obtained by compactifying a ten-dimensional $\mathcal{N} = 1$ supersymmetric theory on a complex three dimensional manifold with $SU(3)$ holonomy. (The general $SO(6) \simeq SU(4)$ holonomy of a real six dimensional manifold is reduced to $SU(3)$, leaving one Killing spinor.) Possible orbifold actions, that lead to four dimensional $\mathcal{N} = 1$ supersymmetry (and that in addition are geometric symmetries of some six-tori[25]) are listed in table 2.2. We will note however, that this classification of supersymmetric orbifold actions is far from being exhaustive. For example

[25]The torus is not completely determined by the symmetry. In general several tori exist for a $\mathbb{Z}_N$ group, that lead to different spectra of the orbifolded theory (cf. [45]).

one can also build products of the above groups. Another possibility are non-abelian orbifolds. In addition to geometric orbifolds, string-theory offers the chance to build asymmetric orbifolds, many of them supersymmetric as well. In these cases, the supersymmetry is recovered in the spectrum. There are also combinations of translations and rotations possible. The Scherk-Schwarz mechanism [46] is an example of such an orbifold. Scherk-Schwarz orbifolds, that generically break supersymmetry admit nevertheless points in parameter space (corresponding to decompactification) where supersymmetry is restored. It is difficult to give a general simple rule which states if supersymmetry exists for an orbifold or not. We also neglected conditions for preserving supersymmetry in the gauge-sector. The restriction to vanishing NSNS field strength H can be weakened. Many recent and some older work elaborated the obstructions in the more general case (cf. [47, 48, 49, 50, 51, 52]). For our purpose the material presented here is sufficient and we will turn to a non-trivial example, featuring asymmetry and the freedom of a $\mathbb{Z}_3$-valued torsion.

2.4 The asymmetric $(T^2 \times T^2)/(\mathbb{Z}_3^{\mathbf{L}} \times \mathbb{Z}_3^{\mathbf{R}})$ orbifold

We explored in section 2.2.2.1 that a two-torus T^2 exists where a $G = \mathbb{Z}_3^{\mathrm{L}} \times \mathbb{Z}_3^{\mathrm{R}}$ subgroup of the $SO(2,2,\mathbb{Z})$ duality group is enhanced to a symmetry such that it can be gauged (i.e. modded out as an orbifold group). This point in moduli space is given by (2.78). It can be rewritten in terms of the mass² matrix (2.60):

$$\frac{M^2}{2} \equiv \frac{1}{2}\begin{pmatrix} \alpha' G^{ik} & -B^i{}_l \\ B_j{}^k & \frac{1}{\alpha'}\left(G - B^2\right)_{jl} \end{pmatrix} = \frac{1}{3}\left(\begin{array}{cc|cc} 2 & -1 & -1/2 & -1 \\ -1 & 2 & 1 & 1/2 \\ \hline -1/2 & 1 & 2 & 1 \\ -1 & 1/2 & 1 & 2 \end{array}\right) \tag{2.103}$$

From this equation we easily see that the compactification scale is of the order of the string scale. We denote the lattice part of the partition function without trace insertions by $\Lambda_{SU(3)^2}$. However this part of the partition function does not factorize into a purely left- and a purely right-moving part. The action on the KK and winding modes $\vec{v} = (m_1, m_2, n_1, n_2)$ is given by (2.72) and (2.76) Besides the $\mathbb{1}$-trace insertion only two other classes of rotations, namely ($\theta(\hat{\theta}^2)$,$\theta^2\hat{\theta}$) and ($\theta\hat{\theta}$, $\theta^2\hat{\theta}^2$), have Eigenvectors on the Narain lattice with Eigenvalue one and can therefore contribute to the lattice-trace. They span the following invariant lattices (the $\vec{r}_i$ are given in the same basis as $M^2/2$ in (2.103)):

$$I_{\theta\hat{\theta}} = I_{\theta^2\hat{\theta}^2} = \{n\vec{v}_1 + m\vec{v}_2 \,|\, n, m \in \mathbb{Z};\ \vec{v}_1 = (1,1,0,1),\ \vec{v}_2 = (1,0,1,0)\} \tag{2.104}$$

$$I_{\theta(\hat{\theta}^2)} = I_{\theta^2\hat{\theta}} = \{n\vec{v}_1 + m\vec{v}_2 \,|\, n, m \in \mathbb{Z};\ \vec{v}_1 = (0,-1,0,1),\ \vec{v}_2 = (1,1,-1,0)\} \tag{2.105}$$

Notifying that the respective normal lattices fulfill the relations:

$$N_{\theta\hat{\theta}} = N_{\theta^2\hat{\theta}^2} = I_{\theta(\hat{\theta}^2)} = I_{\theta^2\hat{\theta}} \qquad N_{\theta(\hat{\theta}^2)} = N_{\theta^2\hat{\theta}} = I_{\theta\hat{\theta}} = I_{\theta^2\hat{\theta}^2} \tag{2.106}$$

and using that $I_{\theta(\hat\theta^2)} \perp I_{\theta\hat\theta}$ (w.r.t. the Lorentzian scalar product Ξ) we derive the useful identities for the (squared) multiplicities $C_{h,g}$ (c.f. (2.101)):

$$\begin{aligned} C_{h=(\theta',\mathrm{Id}),g=(\mathrm{Id},\theta)} &= C_{h=(\mathrm{Id},\mathrm{Id}),g=(\mathrm{Id},\theta)} \\ C_{h=(\mathrm{Id},\theta'),g=(\theta,id)} &= C_{h=(\mathrm{Id},\mathrm{Id}),g=(\theta,\mathrm{Id})} \end{aligned} \tag{2.107}$$

This means that the fix-point (precisely: fix-plane) multiplicity is unaffected by inserting a purely left-moving twist into the trace of a purely right-moving twisted sector. As the fix-point degeneracy in the g-twisted sector is obtained by modular S-transformation of the untwisted sector with g-insertion, we fix again many (in fact: all) prefactors by requiring modular invariance (using (2.101) only indirectly). We would not be able to determine the numbers $C_{h=(\theta',\mathrm{Id}),g=(\mathrm{Id},\theta)}$ by modular transformation of trace inserted untwisted sectors because $k\underset{1}{\square}$ and $h\underset{g}{\square}$ lie in different modular orbits for all $k \in G$.

We get the following bosonic *lattice* partition functions for each T^2:[26]

$$\theta\hat\theta\underset{1}{\square} = \left(\frac{1}{\eta^2(q)}\sum_{\vec v\in\mathbb{Z}^2} q^{\vec v^T S\vec v}\right)\cdot\frac{2\sin(2\pi/3)\eta(\bar q)}{\vartheta\left[{1/2 \atop 1/2+2/3}\right](\bar q)} \tag{2.108}$$

$$\theta^2\hat\theta^2\underset{1}{\square} = \left(\frac{1}{\eta^2(q)}\sum_{\vec v\in\mathbb{Z}^2} q^{\vec v^T S\vec v}\right)\cdot\frac{2\sin(\pi/3)\eta(\bar q)}{\vartheta\left[{1/2 \atop 1/2+1/3}\right](\bar q)} \tag{2.109}$$

$$\theta\hat\theta\underset{1}{\square} = \overline{\theta(\hat\theta^2)\underset{1}{\square}} \qquad\qquad \theta^2\hat\theta^2\underset{1}{\square} = \overline{\theta^2\hat\theta\underset{1}{\square}} \tag{2.110}$$

where $S = \begin{pmatrix} 1 & 1/2 \\ 1/2 & 1\end{pmatrix}$. The q-dependent part in (2.108) multiplies just by a phase $i^{1/6}$ under a modular T-transformation ($\tau \to \tau+1$) but the S-transformed left-moving part is slightly more complicated:

$$1\underset{\theta\hat\theta}{\square} = \frac{1}{\sqrt{3}\eta^2}\sum_{\vec v\in\mathbb{Z}^2} q^{\vec v^T S^{-1}\vec v} \qquad\qquad S^{-1} = \frac{1}{3}\begin{pmatrix} 1 & 1/2 \\ 1/2 & 1\end{pmatrix} \tag{2.111}$$

To calculate further T- and S-transformed partition functions with the help of the Poisson resummation formula (A.14) we have to rewrite the q-dependent part of the above function as a sum over *shifted* lattices:[27]

$$1\underset{\theta^2\hat\theta}{\square} = \frac{1}{\sqrt{3}}\sum_{i=0}^{2}\chi_i \tag{2.112}$$

$$\begin{aligned} \chi_i &= \frac{1}{\eta^2}\left(\sum_{\vec v\in\mathbb{Z}^2} q^{(\vec v+\vec r_i)^T S(\vec v+\vec r_i)}\right) \\ \{r_i | i=0,1,2\} &= \left\{(0,0),\left(\tfrac{1}{3},\tfrac{1}{3}\right),\left(\tfrac{2}{3},\tfrac{2}{3}\right)\right\} \quad (= \mathbb{Z}_3\text{-fix points}) \end{aligned} \tag{2.113}$$

[26] The ϑ-functions are given in appendix A (p. 206). Its modular properties and some identities can be found there as well.

[27] The coordinates of the $\mathbb{Z}_3$-fix points are written in the basis which defines the metric S of the lattice Γ_S.

The sums (2.111) and (2.112) are easily seen to be equal: The KK and winding lattice in (2.111) has one third of the volume of the original lattice Γ_S which is defined by the metric S. This lattice also admits $\mathbb{Z}_3$ symmetry. In addition the fixed-points span a fundamental cell of a $\mathbb{Z}_3$ symmetric lattice *modulo* a lattice vector of Γ_S. This fundamental cell has exactly one third of the volume of the S lattice. Therefore the lattice associated with S^{-1} equals the direct sum:

$$\Gamma_{S^{-1}} = \bigoplus_{r_i \in \mathbb{Z}_3\text{-fix-pts}} \left(\Gamma_S + \vec{r}_i\right) \tag{2.114}$$

Decompositions of the above type appear in compactifications on lattices which are associated with Lie algebras. In our case it is the lattice of the Lie algebra A_2 (or equivalently: $SU(3)$). χ_0 equals the left partition function in (2.108) and multiplies with a phase under a modular T-transformation. In total we can describe the mapping of the so called *characters* χ_i under T and S by matrices[28]:

$$\begin{pmatrix}\chi_0(q)\\ \chi_1(q)\\ \chi_2(q)\end{pmatrix} \xrightarrow{T} e^{-i\pi/6}\begin{pmatrix}1 & 0 & 0\\ 0 & e^{i2\pi/3} & 0\\ 0 & 0 & e^{-i2\pi/3}\end{pmatrix}\begin{pmatrix}\chi_0(q)\\ \chi_1(q)\\ \chi_2(q)\end{pmatrix} \tag{2.115}$$

The S-transformation is a bit more involved. Application of the Poisson resummation formula (A.14) leads to:

$$\frac{1}{\eta^2}\sum_{\vec{v}\in\mathbb{Z}^2} q^{(\vec{v}+\vec{r}_j)^T S(\vec{v}+\vec{r}_j)} \xrightarrow{S} \frac{1}{\eta^2}\frac{1}{\sqrt{\det S}}\sum_{\vec{w}\in\mathbb{Z}^2} e^{i2\pi\vec{r}_j\cdot\vec{w}} q^{(\vec{w})^T S^{-1}\vec{w}} \tag{2.116}$$

The r.h.s. can be split again as a sum according to the decomposition (2.114) of $\Gamma_{S^{-1}}$. The phase in the summation sector (2.116) is then constant in each shifted lattice $\Gamma_S + \vec{r}_k$. It equals: $\exp(i2\pi\vec{r}_k\cdot\vec{r}_j)/3) = \exp(i2\pi(k+j)/3)$. With this information we express the action of S on the characters χ_j by:

$$\begin{pmatrix}\chi_0(q)\\ \chi_1(q)\\ \chi_2(q)\end{pmatrix} \xrightarrow{S} \frac{1}{\sqrt{3}}\begin{pmatrix}1 & 1 & 1\\ 1 & e^{i2\pi/3} & e^{-i2\pi/3}\\ 1 & e^{-i2\pi/3} & e^{i2\pi/3}\end{pmatrix}\begin{pmatrix}\chi_0(q)\\ \chi_1(q)\\ \chi_2(q)\end{pmatrix} \tag{2.117}$$

The above characters χ_i describe a *free* boson compactified on a torus given by the root lattice of $SU(3)$ (or A_2). Because the boson is free, the characters are called *level one*. (In total: “$SU(3)$ characters at level one”.) There are further connections to Kac-Moody algebras which are however not the aim of this thesis. An introduction is given in [6] together with further references. Now we are able to derive the orbits under the modular group. The different orbits under the modular group are marked by boxes with different colours in table 2.3. In total we have *seven* different orbits which are not connected by modular transformations. In principle we could add phases to different orbits. These phases are the discrete torsion introduced at the end of section 2.1. However

[28]We will not explain the relation between general characters and partition functions. A short introduction can be found in [6], chap. 11.

1 1	θ 1	θ^2 1	$\hat\theta$ 1	$\hat\theta^2$ 1	$\theta\hat\theta$ 1	$\theta^2\hat\theta$ 1	$\theta\hat\theta^2$ 1	$\theta^2\hat\theta^2$ 1
1 θ	θ θ	θ^2 θ	$\hat\theta$ θ	$\hat\theta^2$ θ	$\theta\hat\theta$ θ	$\theta^2\hat\theta$ θ	$\theta\hat\theta^2$ θ	$\theta^2\hat\theta^2$ θ
1 θ^2	θ θ^2	θ^2 θ^2	$\hat\theta$ θ^2	$\hat\theta^2$ θ^2	$\theta\hat\theta$ θ^2	$\theta^2\hat\theta$ θ^2	$\theta\hat\theta^2$ θ^2	$\theta^2\hat\theta^2$ θ^2
1 $\hat\theta$	θ $\hat\theta$	θ^2 $\hat\theta$	$\hat\theta$ $\hat\theta$	$\hat\theta^2$ $\hat\theta$	$\theta\hat\theta$ $\hat\theta$	$\theta^2\hat\theta$ $\hat\theta$	$\theta\hat\theta^2$ $\hat\theta$	$\theta^2\hat\theta^2$ $\hat\theta$
1 $\hat\theta^2$	θ $\hat\theta^2$	θ^2 $\hat\theta^2$	$\hat\theta$ $\hat\theta^2$	$\hat\theta^2$ $\hat\theta^2$	$\theta\hat\theta$ $\hat\theta^2$	$\theta^2\hat\theta$ $\hat\theta^2$	$\theta\hat\theta^2$ $\hat\theta^2$	$\theta^2\hat\theta^2$ $\hat\theta^2$
1 $\theta\hat\theta$	θ $\theta\hat\theta$	θ^2 $\theta\hat\theta$	$\hat\theta$ $\theta\hat\theta$	$\hat\theta^2$ $\theta\hat\theta$	$\theta\hat\theta$ $\theta\hat\theta$	$\theta^2\hat\theta$ $\theta\hat\theta$	$\theta\hat\theta^2$ $\theta\hat\theta$	$\theta^2\hat\theta^2$ $\theta\hat\theta$
1 $\theta^2\hat\theta$	θ $\theta^2\hat\theta$	θ^2 $\theta^2\hat\theta$	$\hat\theta$ $\theta^2\hat\theta$	$\hat\theta^2$ $\theta^2\hat\theta$	$\theta\hat\theta$ $\theta^2\hat\theta$	$\theta^2\hat\theta$ $\theta^2\hat\theta$	$\theta\hat\theta^2$ $\theta^2\hat\theta$	$\theta^2\hat\theta^2$ $\theta^2\hat\theta$
1 $\theta\hat\theta^2$	θ $\theta\hat\theta^2$	θ^2 $\theta\hat\theta^2$	$\hat\theta$ $\theta\hat\theta^2$	$\hat\theta^2$ $\theta\hat\theta^2$	$\theta\hat\theta$ $\theta\hat\theta^2$	$\theta^2\hat\theta$ $\theta\hat\theta^2$	$\theta\hat\theta^2$ $\theta\hat\theta^2$	$\theta^2\hat\theta^2$ $\theta\hat\theta^2$
1 $\theta^2\hat\theta^2$	θ $\theta^2\hat\theta^2$	θ^2 $\theta^2\hat\theta^2$	$\hat\theta$ $\theta^2\hat\theta^2$	$\hat\theta^2$ $\theta^2\hat\theta^2$	$\theta\hat\theta$ $\theta^2\hat\theta^2$	$\theta^2\hat\theta$ $\theta^2\hat\theta^2$	$\theta\hat\theta^2$ $\theta^2\hat\theta^2$	$\theta^2\hat\theta^2$ $\theta^2\hat\theta^2$

Table 2.3: Different traces in the partition function. Sectors that belong to the same modular orbit are painted in the same color and style.

higher loop consistency imposes further conditions onto the phases which we have summarized in (2.11) to (2.13) (p. 29). Condition (2.13) forbids nontrivial phases for all modular orbits except the one marked by red (and doubly striped) boxes and the one marked by gray boxes. These two are exactly the orbits that contain elements of the type discussed in (2.107) (i.e. partition functions that are not derived from any σ-untwisted partition function by a modular transformation). We further read off from table 2.3 and the second co-cycle condition (2.12) that these two orbits (each of them containing 24 traces) have complex conjugate torsions ϵ w.r.t. each other. In addition, (2.11) and table 2.3 tells us that $\epsilon^3 = 1$. This leaves two inequivalent choices:[29]

$$\epsilon = 1 \qquad \text{and} \qquad \epsilon = e^{i2\pi/3} \tag{2.118}$$

We will schematically present one orbit explicitly. To be economical with space we introduce the notation of [53]. It is given in appendix A.3. The same notation is used in the discussion of the orientifold of this orbifold in chapter

[29]The choice $\epsilon = e^{-i2\pi/3}$ turns out to be equivalent to $\epsilon = e^{+i2\pi/3}$.

5, section 5.4.1, too. The orbit we choose is the one containing $\theta\hat{\theta}\underset{1}{\blacksquare}$:

$$
\begin{tikzcd}
\theta\hat{\theta}\underset{1}{\blacksquare} \arrow[d, "S"'] & & & & \\
1\underset{\theta\hat{\theta}}{\blacksquare} \arrow[r, "T"] & \theta\hat{\theta}\underset{\theta\hat{\theta}}{\blacksquare} \arrow[r, "T"] & \theta^2\hat{\theta}^2\underset{\theta\hat{\theta}}{\blacksquare} \arrow[d, "S"'] & & \\
& & \theta^2\hat{\theta}^2\underset{\theta^2\hat{\theta}^2}{\blacksquare} \arrow[r, "T"] & \theta\hat{\theta}\underset{\theta^2\hat{\theta}^2}{\blacksquare} \arrow[r, "T"] & 1\underset{\theta^2\hat{\theta}^2}{\blacksquare} \arrow[d, "S"'] \\
& & & & \theta^2\hat{\theta}^2\underset{1}{\blacksquare}
\end{tikzcd}
\tag{2.119}
$$

Explicitly:

$$
\begin{aligned}
\mathcal{T}_{(\theta\hat{\theta})} = \quad & \rho_{00}\Lambda_R\bar{\rho}_{01} \\
& + \rho_{00}\Lambda_W\bar{\rho}_{10} + \rho_{00}\Lambda_W^{(1)}\bar{\rho}_{11} + \rho_{00}\Lambda_W^{(-1)}\bar{\rho}_{12} \\
& \qquad + \rho_{00}\Lambda_W^{(1)}\bar{\rho}_{22} + \rho_{00}\Lambda_W^{(-1)}\bar{\rho}_{21} + \rho_{00}\Lambda_W\bar{\rho}_{20} \\
& \qquad\qquad + \rho_{00}\Lambda_R\bar{\rho}_{02}
\end{aligned}
\tag{2.120}
$$

To obtain the contribution to the torus amplitude we have to integrate the above expression (cf. eq. (2.4)):

$$
T_{(\theta\hat{\theta})} = V_6 \int_{\mathcal{F}} \frac{d^2\tau}{4\tau_2} \overbrace{\int \frac{d^6 p}{(2\pi)^6} e^{-\pi\alpha'\tau_2\|\vec{p}\|^2}}^{(*)} \left(\frac{1}{\eta\bar{\eta}}\right)^4 \mathcal{T}_{(\theta\hat{\theta})}, \quad (*) = \left(\frac{1}{\alpha'\tau_2}\right)^{6/2} \tag{2.121}
$$

$$
= v_d \int_{\mathcal{F}} \frac{d^2\tau}{4(\tau_2)^2} \left(\frac{1}{\sqrt{\tau_2}\eta\bar{\eta}}\right)^4 \mathcal{T}_{(\theta\hat{\theta})} \tag{2.122}
$$

with $v_d \equiv \frac{V_d}{(4\pi^2\alpha')^{d/2}}$ being the regularized d-dimensional space-time volume. The advantage in the definition of the functions ρ_{gh} (cf. A.15, appendix A.3) is their simple modular transformation behaviour (eq. (A.16) and (A.17)). Both the measure $\left(4(\tau_2)^2\right)^{-1}$ and the space-time contribution $\left(\sqrt{\tau_2}\eta\bar{\eta}\right)^{-4}$ of the world-sheet bosons are modular invariant. For completeness we present the complete torus partition function in terms of quantities $\mathcal{T}_{gh}$ that have to be inserted in

ϵ	Spectrum
1	$\mathcal{N}=(2,2)$: Supergravity multiplet
$e^{\pm i2\pi/3}$	$\mathcal{N}=(2,0)$: Supergravity + 21× Tensor multiplet

Table 2.4: Closed-string spectra of the asymmetric $(T^2\times T^2)/(\mathbb{Z}_3^{\mathrm{L}}\times\mathbb{Z}_3^{\mathrm{R}})$ orbifold in dependence of the torsion ϵ.

place of $\mathcal{T}_{(\theta\hat{\theta})}$ into the integral (2.122).[30]

$$\begin{aligned}
\mathcal{T}_{00} &= \frac{1}{9}\Big[\rho_{00}\Lambda_{SU(3)^2}\bar{\rho}_{00} + \boxed{\rho_{01}\bar{\rho}_{01}+\rho_{02}\bar{\rho}_{02}} + \boxed{\rho_{01}\bar{\rho}_{02}+\rho_{02}\bar{\rho}_{01}} \\
&\qquad + \boxed{(\rho_{01}+\rho_{02})\bar{\rho}_{00}\bar{\Lambda}_R} + \boxed{\rho_{00}\Lambda_R(\bar{\rho}_{01}+\bar{\rho}_{02})}\Big] \\
\mathcal{T}_{01} &= \frac{1}{9}\Big[\boxed{\rho_{00}(\Lambda_W\bar{\rho}_{10}+\Lambda_W^{(1)}\bar{\rho}_{11}+\Lambda_W^{(-1)}\bar{\rho}_{12})} + \boxed{\epsilon\rho_{01}(\bar{\rho}_{10}+\bar{\rho}_{11}+\bar{\rho}_{12})} \\
&\qquad + \boxed{\bar{\epsilon}\rho_{02}(\bar{\rho}_{10}+\bar{\rho}_{11}+\bar{\rho}_{12})}\Big] \\
\mathcal{T}_{02} &= \frac{1}{9}\Big[\boxed{\rho_{00}(\Lambda_W\bar{\rho}_{20}+\Lambda_W^{(1)}\bar{\rho}_{22}+\Lambda_W^{(-1)}\bar{\rho}_{21})} + \boxed{\epsilon\rho_{02}(\bar{\rho}_{20}+\bar{\rho}_{22}+\bar{\rho}_{21})} \\
&\qquad + \boxed{\bar{\epsilon}\rho_{01}(\bar{\rho}_{20}+\bar{\rho}_{22}+\bar{\rho}_{21})}\Big] \\
\mathcal{T}_{11} &= \frac{1}{9}\Big[\boxed{9(\rho_{10}\bar{\rho}_{10}+\rho_{11}\bar{\rho}_{11}+\rho_{12}\bar{\rho}_{12})}\boxed{-3\epsilon(\rho_{10}\bar{\rho}_{12}+\rho_{11}\bar{\rho}_{10}+\rho_{12}\bar{\rho}_{11})} \\
&\qquad \boxed{-3\bar{\epsilon}(\rho_{11}\bar{\rho}_{12}+\rho_{12}\bar{\rho}_{10}+\rho_{10}\bar{\rho}_{11})}\Big] \\
\mathcal{T}_{22} &= \frac{1}{9}\Big[\boxed{9(\rho_{20}\bar{\rho}_{20}+\rho_{22}\bar{\rho}_{22}+\rho_{21}\bar{\rho}_{21})}\boxed{-3\epsilon(\rho_{20}\bar{\rho}_{21}+\rho_{22}\bar{\rho}_{20}+\rho_{21}\bar{\rho}_{22})} \\
&\qquad \boxed{-3\bar{\epsilon}(\rho_{22}\bar{\rho}_{21}+\rho_{21}\bar{\rho}_{20}+\rho_{20}\bar{\rho}_{22})}\Big] \\
\mathcal{T}_{12} &= \frac{1}{9}\Big[\boxed{9(\rho_{10}\bar{\rho}_{20}+\rho_{11}\bar{\rho}_{22}+\rho_{12}\bar{\rho}_{21})}\boxed{-3\epsilon(\rho_{10}\bar{\rho}_{22}+\rho_{11}\bar{\rho}_{21}+\rho_{12}\bar{\rho}_{20})} \\
&\qquad \boxed{-3\bar{\epsilon}(\rho_{11}\bar{\rho}_{20}+\rho_{12}\bar{\rho}_{22}+\rho_{10}\bar{\rho}_{21})}\Big]
\end{aligned} \tag{2.123}$$

The remaining integrands for the torus vacuum amplitude (cf. eq. (2.122)) are obtained by complex conjugation:

$$\mathcal{T}_{10}=\overline{\mathcal{T}}_{01} \qquad \mathcal{T}_{20}=\overline{\mathcal{T}}_{02} \qquad \mathcal{T}_{21}=\overline{\mathcal{T}}_{12} \tag{2.124}$$

The spectrum can be read off from the partition function, if one imposes in addition the condition $H=\widetilde{H}$. One also has to distinguish if individual excitations belong to the compact or the space-time part. The spectrum depends on the torsion ϵ. It is listed in table 2.4. We will come back to the $(T^2\times T^2)/(\mathbb{Z}_3^{\mathrm{L}}\times\mathbb{Z}_3^{\mathrm{R}})$ orbifold in chapter 5 where we build an orientifold of the model. Orientifolds will be introduced in the next chapter. In addition to gauging world sheet parity Ω (or a combination $s\Omega$ with a space group element s) they will introduce open

[30] This torus partition function was presented in [53]. We put boxes of the colors used in table 2.3 around the individual contributions, to indicate the modular orbit they belong to.

strings. We will find out that the asymmetric action $\mathbb{Z}_N^{\mathrm{R,\ L}}$ contains D-branes with magnetic fluxes in its orbit. D-branes with electro-magnetic fluxes are the topic of chapter 4.

Chapter 3

Orientifolds

String theories which contain non-orientable world-sheets are called orientifolds. Closed-string theories containing non-orientable diagrams (like the Klein bottle at Euler-characteristic $\chi = 0$) admit tadpoles which lead to inconsistencies. These inconsistencies can often be cured by adding world-sheets with holes and thereby open-strings. At the $\chi = 0$-level these are the Möbius-strip and the Cylinder. By introducing open-strings via D-branes of appropriate number and type, all tadpoles can be eliminated in many cases.[1] Often there exist different D-bane spectra which lead to tadpole cancellation. By this many (attractive) spectra can arise in orientifold models as we will see in the following chapters. After presenting a heuristic definition of a general orbifold, we will enter into the details and consequences like tadpoles, open-strings and associated particle spectra. References and overview articles can be found in the "concluding remarks" on page 86.

3.1 Basic concepts

All of our orientifold models are based on orbifold constructions, where an element $s\Omega$, with Ω the world sheet parity and s an element acting on space time but *not* on the world-sheet, is added to the orbifold group. More generally we define:

Definition 1 *An* **orientifold** *is a string theory, which is obtained by modding out a symmetry group O of the original theory:*

$$O = \overline{G \cup S\Omega} \tag{3.1}$$

G is a group which does not mix left- and right-movers. Ω is the world sheet parity (which exchanges left- and right-movers). S is a set, which defines a

[1]In order to cancel both the NSNS and the RR tadpoles the parent closed-string theory should be supersymmetric. Even though this is not strictly proven, this is likely to be true. There exist also cases (e.g. the Ω orientifold of Type IIB on $T^6/\mathbb{Z}_N$, $N \in 4, 8, 12$ c.f. [40]) where the Klein bottle is supersymmetric but so far no attempt to cancel its tadpoles by D-branes in a supersymmetric manner has been successful and there are hints that this might be impossible.

symmetry of the underlying CFT if it gets multiplied by Ω*. The elements of* S *should not mix the left- and right-moving Hilbert spaces.*

The line over the union indicates algebraic closure. That means that O is a group (and not a half group). As $s\Omega s'\Omega \in O$ we can choose the s-dependent decomposition ($s \in S$):

$$O = G \times \{\mathrm{Id}, s\Omega\} \tag{3.2}$$

where we have redefined G to contain all elements of O which do not mix left- and right-movers. In this thesis we will restrict G to be a toroidal orbifold group. That means that $G = \Gamma \rtimes E$, $E \subset SO(\mathbb{Z})$ (chapter 6 and 7) or $E \subset \Gamma \rtimes \big(SO(\mathbb{Z})_L \times SO(\mathbb{Z})_R\big)$ for the models of chapter 5. An orientifold is in some respect quite similar to an orbifold, but there are also striking differences. In this chapter we will consider several aspects, namely:

- non-oriented spectra (i.e. $S\Omega$-invariant spectra)
- non-orientable world-sheets
- closed-string tadpoles
- D-branes, open-strings and Chan Paton factors

All these aspects are related to each other. We consider the first item which is treated very similar as in pure orbifolds.

3.2 $S\Omega$-invariant closed-string spectra and Klein bottle amplitude

As in an orbifold, we require the states of the orientifold Hilbert-space $|\,\psi\,\rangle \in \mathcal{H}_{\mathcal{O}}$ to be invariant under the whole group O:

$$o\,|\,\psi\,\rangle = |\,\psi\,\rangle \quad \forall\, o \in O \tag{3.3}$$

The closed-string partition function therefore has to include the projection $1/|O| \sum_{o \in O} o$. As we will discuss, this partition function does in general not need to be modular invariant, since the trace taken in the operator formalism (or equivalently the path-integral) does not only correspond to the torus but to the sum of a torus partition function and a so called *Klein bottle* partition function. While the two dimensional world sheet torus can be obtained by dividing the complex plane by a lattice group Γ (cf. eq. (2.5)) the Klein bottle is obtained by dividing the two-torus by a $\mathbb{Z}_2$-symmetry:

$$\Omega: \; z \mapsto -\bar{z} + (1 + i\tau_2/2) \tag{3.4}$$

τ_2 is again the imaginary part of the complex structure, which is purely imaginary, because we require Ω to leave the world sheet metric $h_{\alpha\beta}$ invariant. Note that τ with real parts (i.e. $\tau_1 \neq 0$) would define a $\mathbb{Z}_2$ action leaving the lattice Γ^2 invariant (as a set) if we replace the complex conjugation in (3.4) by a reflection in the τ-direction. However this $\mathbb{Z}_2$ is not a symmetry of the string theory as it changes the world-sheet metric. The Klein bottle with two choices of its fundamental region is depicted in figure 3.1. We have separated the action of Ω (3.4) into the reflection along the $\mathfrak{Re}$-direction and the translation by $1 + i\tau/2$. The first figure corresponds to the loop-channel for the following reason: The string path-integral with string fields integrated over the shaded area (in the upper picture of fig. 3.1) and having the depicted periodicities (Ω from (3.4))[2]

$$X(\tau, \sigma) = \Omega X(\tau, \sigma) = X(\tau + t, \sigma + 1) \tag{3.5}$$

corresponds in operator formalism to the trace-insertion ($q = \exp(-2\pi t)$):

$$\mathrm{KB} = \frac{V_{10}}{(2\pi)^{10}} \lim_{\epsilon \to 0} \int_\epsilon^\infty \frac{dt}{t} Z_{\mathrm{KB}}(q) \tag{3.6}$$

$$Z_{\mathrm{KB}}(q) = \underset{1}{\Omega' \Box} = \frac{1}{|O|} \operatorname{tr} \Omega' e^{-2\pi t(H + \widetilde{H})} \tag{3.7}$$

This follows from the common argument that trace-insertions correspond to periodicity conditions along the time direction in the path-integral picture. Ω' means that in the operator formalism left- and right-movers are exchanged (Ω' : $\sigma \leftrightarrow 2\pi - \sigma$). The shift in the time coordinate $\tau \to \tau + t$ is implicitly included in the trace: (3.7) corresponds to fields in the path-integral which admit the periodicity $X(\sigma, \tau) = \Omega X(\sigma, \tau)$. As the partition function is associated with loop diagrams, we call the associated fundamental region the *loop-channel*. The name *direct channel* is often used, too.

The closed-string which propagates in the loop sweeps out the surface of a Klein bottle. A Klein bottle can be obtained (topologically) by identifying the sides of the shaded rectangle as depicted in figure 3.1. This is topological the same as joining the two ends of a cylinder as described in the sequence of figure 3.2. We have painted a grid (even though the Klein bottle admits of course no holes) to make the situation more transparent. The Klein bottle has inevitably self-intersections when embedded into three dimensions. Equation (3.5) can easily be generalized to cases where Ω is combined with some element g of G and $s \in S$ from definition 1. The closed-string can also have boundary conditions twisted by h and g in the τ- resp. σ-direction on the underlying torus as depicted in figure 3.1. However consistency of the $\mathbb{Z}_2$ involution Ω (3.4) puts some constraints on the combination of $h, g, k \in G$ and $s \in S$ if we consider the trace-insertion $ks\Omega$: [3]

$$(ks\Omega)^2 = h \ , \ (ks\Omega g)^2 = h \tag{3.8}$$

$$\Longrightarrow h = (ks\Omega)^2 \ , \ g\, ks\Omega\, g = ks\Omega \tag{3.9}$$

The last equivalence in (3.8) defines exactly the fundamental group π_1 of the Klein bottle if we take g and $ks\Omega$ as its generators. In addition to these relations (which are realized on the fields) the derivatives $\partial_\pm$ (cf. (2.26)) get exchanged:

$$\Omega \partial_\pm = \partial_\mp \Omega \tag{3.10}$$

$$\Rightarrow ks\Omega\partial_\pm X(\tau, \sigma) = ks\partial_\mp \Omega X(\tau, \sigma) = ks\partial_\mp X(\tau + t, -\sigma + 1) \tag{3.11}$$

[2]The world sheet time τ is assumed to be along the $\mathfrak{Im}$ direction while the world sheet coordinate σ is along the $\mathfrak{Re}$-axis.

[3]Ω can be omitted if g, k and s act left-right symmetric.

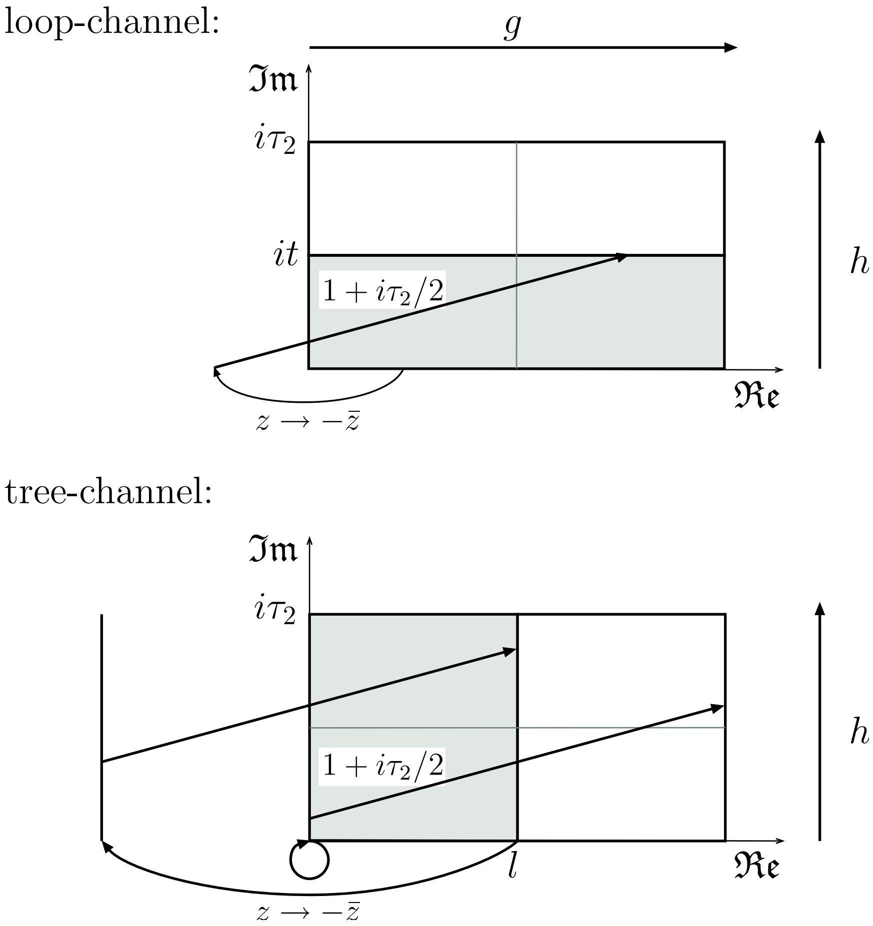


Figure 3.1: The periodicities of the Klein bottle embedded in the underlying torus (-cell) and the two fundamental regions of the Klein bottle (shaded areas).

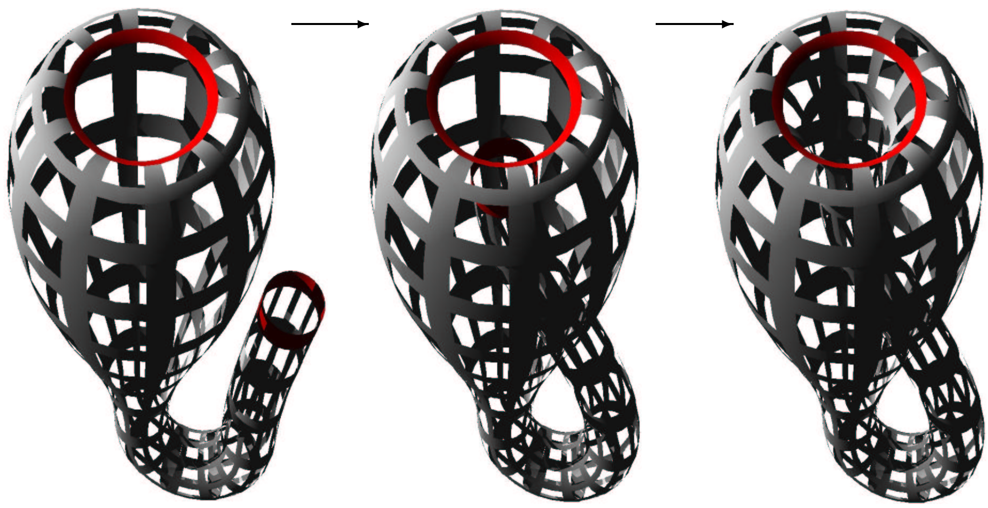


Figure 3.2: Construction of the Klein bottle by joining two ends of a cylinder in the way depicted.

One has to distinguish between world sheet time τ and the modular parameter $it = i\tau_2/2$ of the underlying Torus T^2. In the *tree-channel* (sometimes called: *transverse channel*)

$$X(\tau,\sigma) = ks\Omega X(\tau,\sigma) \quad \Rightarrow \begin{cases} X(\tau,\sigma)\big|_{\sigma=\frac{1}{2}} & = ksX(\tau+t,\sigma)\big|_{\sigma=\frac{1}{2}} \\ \partial_\pm X(\tau,\sigma)\big|_{\sigma=\frac{1}{2}} & = ks\partial_\mp X(\tau+t,\sigma)\big|_{\sigma=\frac{1}{2}} \end{cases} \tag{3.12}$$

The line at $\sigma = 0$ is mapped to the line at $\sigma = 2l = 1$ in a similar way. If we take the twist $X(\tau,\sigma+1) = gX(\tau,\sigma)$ on the underlying torus into account, we get analogously:

$$\begin{aligned} X(\tau,\sigma)\big|_{\sigma=0} &= ksX(\tau+t,\sigma=1) = ksgX(\tau+t,\sigma)|_{\sigma=0} \\ \partial_\pm X(\tau,\sigma)\big|_{\sigma=0} &= ks\partial_\mp X(\tau+t,\sigma=1) = ksg\partial_\mp X(\tau+t,\sigma)\big|_{\sigma=0} \end{aligned} \tag{3.13}$$

We notice that in the case of $k = h = g = s = \mathrm{Id}$ the boundary conditions (3.12) and (3.13) define topologically what is called a *cross-cap* as they identify opposite points on the ends of a cylinder. The formulæ (3.12) and (3.13) give a hint to an alternative calculation of the Klein bottle amplitude. We could as well calculate the path-integral with the fields argument lying in the fundamental region which we called the tree-channel in figure 3.1. This channel is depicted in figure 3.3 as a cylinder with two cross-caps (twisted by ks and ksg). These fields have to fulfill the periodicity (or boundary) conditions described by (3.12) and (3.13). The tree-channel corresponds to h-twisted closed-strings which travel from the $\tau = 0$ line to $\tau = l$ if we Weyl-rescale the world sheet by $\lambda = 1/(2t)$, such that $\tau_2 = 2t \mapsto 1$ and the propagation length $l = 1/2$ rescales accordingly to $l = 1/(4t)$. The rescaling prescription is summarized in table 3.2 (p. 64) together with the corresponding values for the annulus and Möbius strip which we will introduce in section 3.3. In the operator formalism the tree-channel amplitude is described as a transition amplitude between two *cross-cap states* $|\mathcal{C}(ks\Omega)\rangle$ and $|\mathcal{C}(ks\Omega g)\rangle$ which obey the boundary condition (3.12) resp. (3.13):[4]

$$\widetilde{\mathrm{KB}}_{(ks,ksg)} = \int_{\mathbb{R}^+} dl\, \big\langle \mathcal{C}(ks) \big| e^{-2\pi l (H+\widetilde{H})_{h\text{-twisted}}} \big| \mathcal{C}(ksg) \big\rangle \text{ with } h = (ks\Omega)^2 \quad (3.14)$$

$$\begin{aligned} \big(X(\tau=0,\sigma) - aX(\tau=0,\sigma+1/2)\big)_{h\text{-twisted}} |\mathcal{C}(a)\rangle = 0 \\ \big(\partial_\pm X(\tau=0,\sigma) + a\partial_\mp X(\tau=0,\sigma+1/2)\big)_{h\text{-twisted}} |\mathcal{C}(a)\rangle = 0 \end{aligned} \quad (3.15)$$

We expect that the above condition (3.15) can only be obeyed if we restrict the coordinate field $X(\tau,\sigma)$ and its derivatives $\partial_\pm X(\tau,\sigma)$ to the annihilation- (or: positive frequency-) part. The same will be the case for the boundary states which correspond to D-branes that will be introduced in section 3.3.

The boundary conditions in (3.15) are defined at $\tau = 0$. However the time evolution operator $\exp\big(-2\pi l(H+\widetilde{H})\big)$ appearing in the integrand of (3.14) maps one of them to $\tau = l$. The boundary conditions do not determine the normalization of the cross-cap states. They do not fix the phase, too. The normalization is fixed if we compute the amplitude in loop-channel, which has to lead to an identical result, since we are choosing a different but equivalent fundamental region of the Klein bottle:

$$\mathrm{KB}_{(ks,ksg)} = \frac{V_{10}}{(2\pi)^{10}} \lim_{\epsilon\to 0} \int_\epsilon^\infty \frac{dt}{t} Z_{\mathrm{KB}_{(ks,ksg)}}(q) \quad (3.16)$$

$$Z_{\mathrm{KB}_{(ks,ksg)}}(q) = \frac{1}{|O|} \operatorname{tr} ks\Omega' e^{-2\pi t (H+\widetilde{H})_{g\text{-twisted}}} \quad (3.17)$$

Summing over all allowed twists h and g and inserting all allowed trace-insertions containing $k\Omega' s$ gives the complete non-orientable closed-string contribution to the one-loop amplitude:[5]

$$\begin{aligned} \widetilde{Z}_{\mathrm{KB}} &= \sum_{\substack{g,k\in G \\ g\,ks\Omega\, g = ks\Omega}} \widetilde{Z}_{\mathrm{KB}_{(ks,ksg)}} && (3.18) \\ &= \sum_{\substack{g,k\in G \\ g\,ks\, g = ks}} \big\langle \mathcal{C}(ks) \big| e^{-2\pi l (H+\widetilde{H})_{h^2\text{-twisted}}} \big| \mathcal{C}(ksg) \big\rangle && (3.19) \\ &= \sum_{h\in G} \big\langle \mathcal{C}|s;h \big| e^{-2\pi l (H+\widetilde{H})_{h^2\text{-twisted}}} \big| \mathcal{C}|s;h \big\rangle && (3.20) \end{aligned}$$

where we have defined the complete cross-cap state in the h-twisted tree channel (with orientifold projection $s\Omega$, $s \in S$) by:

$$|\mathcal{C}|s;h\rangle \equiv \sum_{\substack{a\in G \\ (as)^2 = h}} |\mathcal{C}(as)\rangle \quad (3.21)$$

[4] A "cross-cap state" is a special kind of boundary state. "Cross-cap states" only appear in orientifolds, whereas general boundary states are present in any theory with open-strings.

[5] The tilde over Z indicates that its modular parameter is expressed as $q = \exp(-2\pi l)$. In the following we will skip the Z if it is clear from the context that we consider only the integrand (which is the partition function) but not the whole expression (which is the amplitude).

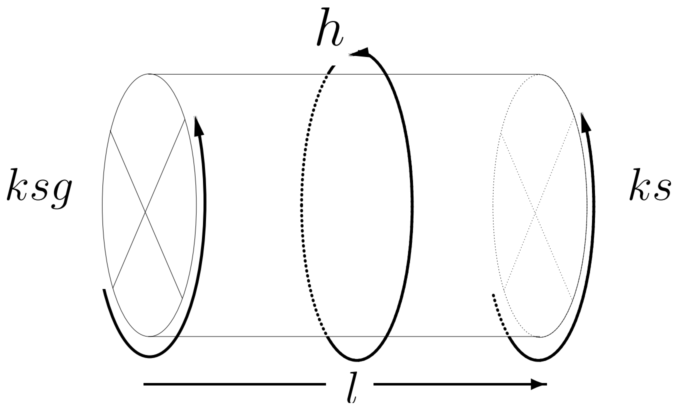


Figure 3.3: The Klein bottle in tree-channel. The twist h of the closed-string is restricted to: $h = (ks\Omega)^2 = (ks\Omega g)^2$.

In (3.20) the Klein bottle tree channel integrand $\widetilde{Z}_{\mathrm{KB}}$ is written as a sum of *complete squares* with a product defined by $(a, b) = \langle a | \exp -2\pi l(H + \widetilde{H}) | b\rangle$. (Taking into account that a) the closed-string propagator does not mix differently twisted sectors and that b) the scalar product of states in differently twisted sectors vanishes, we can write (3.20) as a single complete square.) The situation is very similar to the case of orbifolds: In orbifolds the knowledge of traces in the untwisted sectors is sufficient to determine many twisted sectors by modular transformations (cf. eq. (2.9)). In the orientifold we can determine cross-cap state normalizations from loop amplitudes in the untwisted sector of the loop-channel by considering amplitudes of the form:

$$\widetilde{\widetilde{\mathrm{KB}}}_{(ks,ks)} = \mathrm{KB}_{(ks,ks)} \tag{3.22}$$

for all $k \in G,\ s \in S$. This leaves still a phase for the cross-cap state which we can not determine in this way. In principle we have shown, how one can calculate the one-loop vacuum amplitudes for the closed-string sector of an orientifold. The two diagrams are the torus from the orbifold theory and the Klein bottle amplitude which is unique to orientifolds. We should be aware that the Klein bottle amplitude describes a geometrically closed surface only on the orbifold space and in general not in the ambient space. In addition for the Klein bottle amplitude to describe a closed world-sheet in space time, one has to require $S = \{\mathrm{Id}\}$. The Euler character for general surfaces with h handles, b boundaries and c cross-caps is given by:

$$\chi = 2 - 2h - b - c, \tag{3.23}$$

In this sense both the torus and the Klein bottle are $\chi = 0$ amplitudes, i.e. first order in perturbation theory. Even though we can formally calculate the amplitude we have so far neglected one important problem which we will treat in the following section.

3.2.1 Closed-string tadpoles

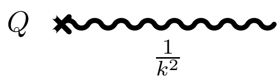
We recall that by choosing an appropriate fundamental region $\mathcal{F}_0$ for the integration region of τ (cf. fig. 2.2) we can avoid singularities in the integrand of the torus amplitude (2.4). However the modular group of the Klein bottle is trivial and we will in general encounter a singularity (or multiple ones) in the integrand of (3.16) for $t \to 0$.[6] This or these divergences are mapped to a divergence (or multiple divergences) $\propto dl\, q^0 \times$ (volume factors) for $l \to \infty$ in tree-channel. There could be multiple divergencies with different volume dependencies (e.g. in orientifolds of (toroidal) orbifolds). However, we will say "the" divergence in the following. We can give a physical interpretation of the divergence if we observe that the $l \to \infty$, q^0 part of the tree-channel corresponds to massless closed-string states traveling an infinite distance in space (cf. fig. 3.3). In field theory there is an analogous phenomenon called *tadpole*. We take as a simple example the following action for a real scalar field ϕ:

$$\int d^d x \left(\tfrac{1}{2}\partial_\mu\phi\partial^\mu\phi + Q\phi\right) \tag{3.24}$$

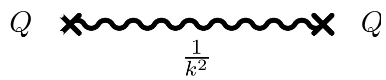
The equation of motion is:

$$\partial_\mu\partial^\mu\phi = Q \tag{3.25}$$

If we expand around $Q = 0$ we will encounter Feynman diagrams like



and



(3.26)

Both diagrams have divergencies at vanishing momentum $k^2 \to 0$. If we rewrite the propagator $1/k^2$ as

$$\frac{1}{k^2} = \int_0^\infty dl\, e^{-k^2 l} \tag{3.27}$$

we inspect that this divergence originates from huge times l in the time evolution operator $\exp(-k^2 l)$. We could have avoided the divergence if we would have expanded around the true vacuum (3.25). The divergence in the Klein bottle amplitude can also be traced back to an ill defined vacuum. There are several plausible solutions to the problem: one is to continously deform the values of background fields (in a space time dependent manner) such that the resulting theory is tadpole-free and consistent. This would be the so called *Fischler-Susskind mechanism* [54, 55] or a generalization thereof. However there exist situations where this continuous deformation does not remove the tadpole. The Feynman graphs in (3.26) would be the same if we have introduced (uncanceled) background charges Q to which a gauge field couples (represented by the wiggled line in the diagram). These charges can be of topological nature and they can not be canceled by deforming the background continously. This is usually the case in orientifolds. As an example we take a closer look at the 10-dimensional Type I string which can be derived from Type IIB superstring by constructing the orientifold with $O = \{1, \Omega\}$. The spectra of both Type IIB and Type I

Type IIB	bosons	
	NS-NS metric g_{ij}, 2-form B_{ij}, dilaton ϕ	R-R scalar χ, 2-form B'_{ij}, self-dual 4-form D_{ijkl}
	fermions	
	NSR gravitino $\psi_{i\dot{a}}$	RNS gravitino $\tilde{\psi}_{j\dot{b}}$
Type I	bosons	
	NS-NS Metric g_{ij}, dilaton ϕ	R-R 2-form B'_{ij},
	fermions	
	1 gravitino $\psi_{j\dot{b}}$	

Table 3.1: Massless closed-string spectra of Type IIB and Type I

(closed-strings only) superstring are listed in table 3.1. The spectrum of the (un-oriented) Type I theory is obtained by projecting onto Ω-invariant states. When one computes the tree-channel KB amplitude a tadpole stemming from an NSNS state and another tadpole from the RR sector shows up. Both tadpoles have the same magnitude but opposite sign. The sum of both vanishes of course, since the tree-channel amplitude can be rewritten as an integral over the partition function which has to vanish due to supersymmetry. However both tadpoles are associated with physically distinguishable fields, which lead to equations of motion for both fields. The tadpole means that both field equations are not fulfilled in our background. The 10-dimensional superstring has a peculiar problem in the RR sector. The tadpole is Lorentz invariant. It should be transmitted by a scalar field (or its Poincaré dual: in 10 dim. this is an 8-form). However no scalars exist in the (perturbative) RR sector of Type I (table 3.1). But a 10-form potential C_{10} would give raise to a Lorentz invariant tadpole, if it appears via

$$S_{\text{C.S.}} = \mu_{10} \int d^{10}x \, C_{10} \tag{3.28}$$

in the action. The action would be invariant under the gauge transformation $C_{10} \mapsto C_{10} + d\Lambda_9$. So C_{10} is in principle charged under this $U(1)$ symmetry. Dimensional reasons forbid a kinetic term $F_{11} = dC_{10}$. So the eom takes the form $\mu_{10} = 0$. This can not be fulfilled because μ_{10} is a constant. However the action (3.28) should be interpreted as the coupling of the form C_{10} to a 10-dimensional object. The action (3.28) is of topological type. It is a so called *Chern Simons action*. We could now add other objects charged under C_{10} s.th. the total action for C_{10} vanishes. These objects are D9-branes in the case of the Type I string. Thereby we automatically introduce open-strings and open-string diagrams. Before we will have a closer look at the open-string sector of

[6] However there exist examples with finite KB amplitude.

orientifolds, we will make a comment about other possible tadpoles.
The eoms of a $p+1$ form field strength H_{p+1} can be written as a generalization of the Maxwell equations in 10 space-time dimensions:

$$dH_{p+1} = *J_{8-p} \qquad\qquad d * H_{p+1} = *J_p \tag{3.29}$$

J_{8-p} is the magnetic source and J_p the electric one. (The J's depend in general on other fields of the theory as well.) For $p = 10$ like in Type I theory no field strength exists. As a consequence the associated total charge J_{10} has to vanish pointwise. For $p < 10$ and a spacetime without boundaries the eoms imply that the following integral vanishes by Stoke's theorem:

$$\int\limits_{\mathcal{M},\, \partial\mathcal{M}=\emptyset} *J_k = 0 \tag{3.30}$$

In the low energy field theory limit the orientifold projection in the closed-string sectors is described by adding expressions of the type:

$$S_{\text{C.S.}} = Q_q \int_{O_q} \sqrt{\frac{\hat{\mathcal{L}}(\mathcal{R}_T/4)}{\hat{\mathcal{L}}(\mathcal{R}_N/4)}} \wedge \sum_{p\in\text{RR}} C_p \tag{3.31}$$

These actions are of Chern Simons type and therfore topological. In (3.31) we have introduced the Hirzebruch polynomial:

$$\hat{\mathcal{L}}(\mathcal{R}) = 1 + \frac{p_1(\mathcal{R})}{3} + \frac{p_2(\mathcal{R}) - \big(p_1(\mathcal{R})\big)^2}{45} + \dots \tag{3.32}$$

with p_i the i^{th} Pontrjagin class. $\mathcal{R}_T$ and $\mathcal{R}_N$ are the pull-backs of the curvature two-form to the tangent and normal bundle of the subspace O_q. O_q is called the *orientifold q plane* (short: Oq plane). The Oq planes are in direct correspondence to cross-cap states $|\mathcal{C}\rangle$: In geometric orientifolds, the q-planes are q dimensional subspaces which are left point-wisely invariant by the element gs ($g \in G,\ s \in S$) appearing in the definition (3.15) of the cross-cap state $|\mathcal{C}(gs)\rangle$ ($O_q(gs) = \{x \mid x \in \mathcal{X},\ x = gs\, x\}$). If we are able to normalize the action (3.31) correctly, we can in principle determine the O-plane charge without doing the explicit CFT calculation. This program has been carried out in [43] for a class of orientifolds descending from supersymmetric Type IIA closed-string theories (compactified on Calabi-Yau spaces). In these models the set S consists of a $\mathbb{Z}_2$-involution $\bar{\sigma}$ leaving invariant a $d/2$-dimensional subspace of the CYd space. In addition the $\bar{\sigma}$ considered in this publication exchanges forms with complex conjugate forms of the CY space (i.e. it maps a (p,q)-form to a (q,p)-form). The resulting O-(hyper)planes are *special Lagrangian submanifolds* (short: sLags).[7] In the CFT description this leads in general to a supersymmetric closed-string sector (if one starts with a supersymmetric string theory). Orientifolds of this kind have been also considered in [56, 57, 58, 59]. In [3] we investigated a $\mathbb{Z}_4$ orientifold of this kind in four space-time dimensions which is supersymmetric

[7] A definition of a sLag is given in section 6.5.1.

	Klein bottle	Annulus	Möbius strip
loop-channel (direct channel)	t	t	t
tree-channel (transverse channel)	$\frac{1}{4l}$	$\frac{1}{2l}$	$\frac{1}{8l}$

Table 3.2: Relation between the parameters t and l in the loop- and tree-channel.

in both closed- and open-string sectors and which is in addition chiral in the open-string sector. Furthermore, it has other phenomenologically appealing features. We will return to this orientifold in chapter 7. The NSNS tadpole can be derived from a *Dirac Born Infeld*-type action which for the O-plane is proportional to the volume of the hyperplane:

$$S_{\mathrm{DBI}} = T_q \int_{O_q} e^{-\Phi} \sqrt{-\det G} \tag{3.33}$$

G is the pullback of the space-time metric to the Oq-plane, Φ the Dilaton and the constant T_q is the so called *tension* of the Oq-plane. The action 3.33 is *not* topological in contrast to the Chern Simons action (3.31). In principle the NSNS tadpole can be removed by a continuous deformation of the background. However this deformation can lead to a degenerate space (e.g. zero or infinite volume in compactifications).

3.3 Open-strings

Open-strings are very similar to closed-strings. However open-string diagrams have boundaries. Their Lagrangian can be written as a sum of a bulk and of a boundary Lagrangian.[8] The bulk Lagrangian is the same for closed- and open-strings. Though it is only integrated from $\sigma = 0$ to $\sigma = \pi$. In other words: The open-string has half the length of a closed-string. This normalization is important for the comparison of different closed- and open-string amplitudes and has direct impact on the tadpole cancellation conditions. We will have a closer look at the details of the boundary terms in the next chapter. There also the open-string Lagrangian can be found. Here we only summarize some results. The boundary conditions for the open-string take the form:

$$\partial_+ X(\tau,\sigma) = V_{(i)} \partial_- X(\tau,\sigma)\,,\ \sigma \in \partial\mathcal{M}_i \tag{3.34}$$

$\mathcal{M}_i$ is the i^{th} connected part of the world sheet boundary. We can associate it with an object usually called a *D-brane*. Sometimes the word D-brane implies more: e.g. a special amount of space time supersymmetry which is preserved by this kind of boundary condition. In order to consider space-time supersymmetry one has to generalize these boundary conditions to the fermionic sector of the world-sheet fields (if one applies RNS formalism, cf. section 3.3.4). We will consider Chan-Paton degrees of freedom (dofs) later which lead to a slightly

[8]We consider for simplicity only string theories which can be described by Lagrangians.

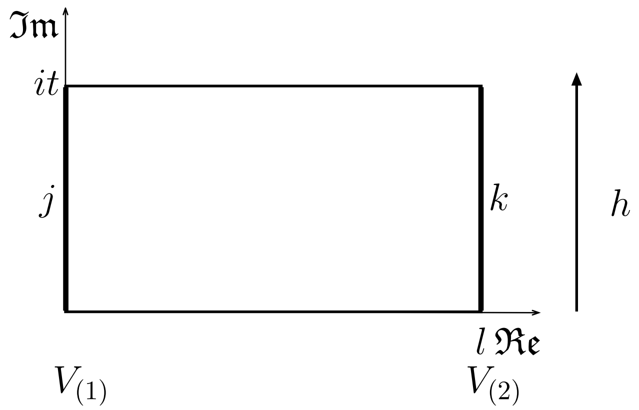


Figure 3.4: The periodicities of the cylinder with boundary conditions $V_{(1)}$ and $V_{(2)}$ as well as Chan Paton labels j, k.

different characterization of a D-brane. The $V_{(i)}$ are matrices that specify the boundary conditions. In all of our concrete examples they are orthogonal matrices. As will be shown in chapter 4, electric fields coupling to the boundary will invoke $V_{(i)} \in SO(1, d-1)$. There the concrete form of the $V_{(i)}$ can be found (cf. eq. (4.19), p. 91).

Equation (3.34) does not include zero-modes. Zero-modes are nevertheless important as they govern multiplicities in the spectrum. We will not enter in the details here. In the examples presented in this work the zero-mode contribution can be derived consistently. We are especially interested in $\chi = 0$ diagrams. According to eq. (3.23) the remaining diagrams are:

- Cylinder (or annulus) ($b = 2$, $h = c = 0$)
- Möbius strip ($b = c = 1$, $h = 0$)

The annulus is depicted in figure 3.5. We have already included possible twists in the boundary conditions which appear if we consider the CFT on this cylinder. The cylinder is not obtained by modding out a torus CFT by a finite group. The world sheet of a torus is simply a square with two opposite sides identified in an orientation preserving way. (In figure 3.4 the horizontal lines are identified.) Therefore the torus has only one fundamental region. The complex structure is however fixed to be purely imaginary due to the boundary conditions (3.34). Nevertheless, there exists a notion of a loop- and a tree-channel as well. The loop-channel corresponds to open-strings of length π with end-points k and j (cf. fig. 3.5) propagating in a loop. In the path-integral the corresponding fields are periodic in the world-sheet τ-direction up to a twist $h \in G$. This implies that the boundary conditions are invariant as well as the CP-labels (in figure 3.5: j and k), which we consider in section 3.3.1. The tree-channel is just another parameterization of the one cylinder (fig. 3.4). It is interpreted as closed-strings (of length 2π) propagating from (closed-string) world sheet time $\sigma = 0$ to $\sigma = l$. The correspondence between tree- and loop-channel parameters is computed in the same way as in the Klein bottle by Weyl rescaling with

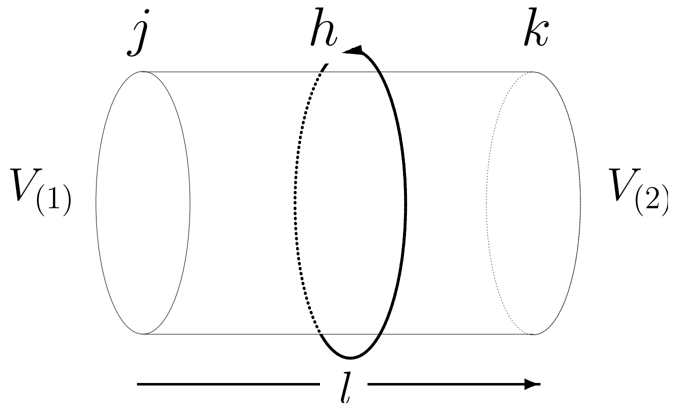


Figure 3.5: The cylinder diagram. The boundary conditions are represented by $V_{(1)}$ and $V_{(2)}$. The twist in the closed-string channel is denoted by h. As it can be also viewed as a loop diagram, we have included the Chan Paton indices j and k. These have to fulfill the condition $h(j) = j$ and $h(k) = k$. h acts differently on the left and right CP index (cf. text).

$\lambda = 1/t$ such that $l = 1/2 \mapsto l = 1/(2t)$ (taking into account the length of the open-string). This result is listed in table 3.2. In analogy to cross-cap states the so called *boundary states* are defined by (up to normalization and phase):

$$\left(\partial_+ X(\tau = 0, \sigma) + V \partial_- X(\tau = 0, \sigma)\right)_{h\text{-twisted}} |\mathcal{D}(V); h\rangle = 0 \tag{3.35}$$

We also have to impose analogous conditions on the zero-modes. In addition the boundary condition specified by V (c.f. (3.34)) has to be invariant under h (c.f. section 3.3.1.1, eq. (3.44)). The world-sheet coordinates in (3.35) correspond to the tree-channel parameterization.

3.3.1 Chan Paton factors and gauge symmetries

In fig. 3.5 we have already included so called *Chan Paton* indices (short: CP-indices). Their name is devoted to its inventors [26]. It has been observed that additional dofs can be added to the endpoints of open-strings. In general a string vacuum (on which the perturbative Fock-space is built by creation operators) looks like:

$$|\theta(i, j); a; b\rangle \tag{3.36}$$

a and b are the CP-indices associated with the $\sigma = 0$ and $\sigma = \pi$ ends of the open-string, $\theta(i, j)$ labels sectors which are given by boundary conditions including (3.34) and conditions for the zero modes as well. The first entry in θ refers to the $\sigma = 0$ point (which we call the left end-point), the other to the $\sigma = \pi$ boundary (the right end-point) of the open-string, in analogy with the CP indices. We have assumed here that the $V_{(i)}$ of (3.34) do not act on the CP degrees of freedom. However we do not want the states (3.36) to factorize. This means that the CP-Hilbert spaces are in general distinct in different sectors

$\theta(i,j)$ of the ambient theory:[9]

$$\mathcal{H} = \bigoplus_{i\in \mathrm{U}} \bigoplus_{j\in \mathrm{U}} \big(\mathcal{H}^{\mathrm{L}}_{(i)} \otimes \mathcal{H}^{\mathrm{R}}_{(j)}\big) \otimes \mathcal{H}_{\theta(i,j)} \tag{3.37}$$

$\mathcal{H}^{\mathrm{L}}_{(i)}$ and $\mathcal{H}^{\mathrm{R}}_{(j)}$ correspond to the CP-Hilbert space associated with the left and with the right string end-point. On the other hand the string end-points are associated with (a set of) D-branes obeying the boundary condition (3.34). The left and right Hilbert spaces are not independent. In order to define a consistent perturbation theory, left and right spaces have to be identified. For string theory with CP dofs to be consistent, the corresponding vertex operators of the states (3.36) have to exist. The interactions described with the help of these operators have also to be consistent. This is possible with the Hilbert space structure given by (3.37) (though we give no rigorous proof). We assume that $\mathcal{H}^{\mathrm{L/R}}_{(i)}$ is a $n(i)$ dimensional vector spaces over $\mathbb{C}$ ($n < \infty$). The Chan Paton states can be represented by $n \times m$ matrices $\Lambda^{(i,j)}_{ab} \in \mathbb{C}^{n(i)} \otimes \mathbb{C}^{m(j)}$. The inner product induced by the hermitian product on $\mathbb{C}^{n(i)}$ and $\mathbb{C}^{m(j)}$ takes the form:

$$\big\langle \theta(i,j); \Lambda^{(i,j)} \big| \theta(k,l); \Lambda^{(kl)}_2 \big\rangle = \delta^{ik}\delta^{jl}\ \mathrm{tr}\big(\Lambda^{(i,j)\dagger}_1 \Lambda^{(k,l)}_2\big) \tag{3.38}$$

Different states of a quantum theory should be normalized. Furthermore, the (overall) phase of the (complete) state is not important for the amplitudes that we measure. For identical left and right Hilbert spaces (which we require in the case of identical boundary conditions $(i,j) = (i,i)$) the set of hermitian matrices with the following normalization:

$$\mathrm{tr}\left(\Lambda^{(i,i)}_{(a)} \Lambda^{(i,i)}_{(b)}\right) = \delta_{ab} \qquad a,b = 1 \ldots \big(n^{(i)}\big)^2 \tag{3.39}$$

forms a complete set of states in the $n^{(i)} \times n^{(i)}$-dimensional Hilbert space $\mathcal{H}^{\mathrm{L}}_{(i)} \otimes \mathcal{H}^{\mathrm{R}}_{(i)}$. The $\mathbb{C}$-vector space of these matrices forms a representation space for the irreducible adjoint representation of $U(n)$. The adjoint representation of $SU(n)$ acts however only reducibly on this space.

In scattering amplitudes two such CP matrices Λ get multiplied if they belong to adjacent string endpoints of open-string vertex operators (no sum over upper indices which specify the boundary conditions):

$$\mathcal{A}_{1,2,\ldots n} \propto \mathrm{tr}\big(\Lambda^{(i,j)}_1 \Lambda^{(j,k)}_2 \ldots \Lambda^{(l,i)}_n\big) \tag{3.40}$$

By this form we have implicitly identified left with right CP-Hilbert spaces: $\mathcal{H}^{\mathrm{L}}_{(i)} \sim \mathcal{H}^{\mathrm{R}}_{(i)}$. Of course one will also encounter sums over expressions of the type (3.40) if one considers permutations of vertex operators. Inner boundaries (i.e. without external string states attached) get multiplied just by:

$$\sum_i \sum_a \mathrm{tr}\,\Lambda^{(i,i)}_{(a)} \qquad i\text{: inner boundary} \tag{3.41}$$

[9]By ambient we mean that we have not performed any orbifold projection so far. As we will see in section 3.3.1.1, the structure of $\mathcal{H}$ might change on orbifolds.

The expressions (3.40) and (3.41) are clearly invariant if we perform the following combined transformations:

$$\begin{aligned} \Lambda^{(i,j)} &\mapsto \Lambda^{(i,j)} U^{(j)} \\ \Lambda^{(j,k)} &\mapsto \left(U^{(j)}\right)^{-1} \Lambda^{(j,k)} \end{aligned} \tag{3.42}$$

$U^{(j)}$ is associated with the j^{th} stack of D-branes. To be a symmetry of the full quantum theory, $U^{(j)}$ should preserve the inner product (3.38) (or at least its modulus) so that it is restricted to be a unitary (or anti-unitary) map of dimension $n(j)$. (We restrict ourself to the unitary case for simplicity in which case $U^{(j)}$ is described by a unitary matrix.) Note that in this sense the left Hilbert space $\mathcal{H}^{\mathrm{L}}_{(i)}$ transforms in the *complex conjugate* representation in comparison to the right Hilbert space $\mathcal{H}^{\mathrm{R}}_{(i)}$:

$$\begin{array}{rcl} \mathcal{H}^{\mathrm{L}}_{(i)} \otimes \mathcal{H}^{\mathrm{R}}_{(j)} \ \ni v^{\mathrm{L}} \otimes w^{\mathrm{R}} & \xrightarrow{U^{(i)} \times U^{(j)}} & \left(U^{-1}_{(i)} v\right) \otimes \left(U^{T}_{(j)} w\right) \\ \Lambda^{(i,j)} & \longrightarrow & U^{-1}_{(i)} \Lambda^{(i,j)} U_{(j)} \end{array} \tag{3.43}$$

It turns out that strings in the sector $\theta(j,j)$ give rise to vectors[10]. The global $U(n(j))$ symmetry is then lifted to a local one, i.e. a *gauge symmetry*. Strings in sectors $\theta(i,j)$ with $i \neq j$ usually contain massless matter as moding and vacuum energy are lifted (cf. chapter 4). In addition Lorentz symmetry gets broken by this kind of boundary conditions. The open-string massless matter states then transform in the bifundamental $n(i) \times \bar{n}(j)$ of $U(n(i)) \times U(n(j))$. It is natural to ask whether the Chan-Paton Hilbert-space $\mathcal{H}^{\mathrm{L}} \otimes \mathcal{H}^{\mathrm{R}}$ which we assumed to be a tensor product $\mathbb{C}^n \otimes \mathbb{C}^m$ might be projected to a smaller space, thereby reducing the $U(n) \times U(m)$ to some smaller symmetry group. It turns out that this is indeed possible. However not all gauge groups are possible and there are constraints. Related with these issues is the question if unitarity conditions like *factorization of tree level amplitudes* are respected. For unitary gauge groups these conditions are obeyed. Unitary gauge groups or products thereof are the only perturbative gauge-symmetries of oriented closed string Type II spectra.

We will now consider the case where additional symmetries are modded out, such that the Hilbert space $\mathcal{H}$, eq. (3.37) gets modified.

3.3.1.1 Open-strings on orbifolds

In building the orbifold we have first to symmetrize our set of boundary conditions represented by the $V_{(i)}$ i.e. we consider all boundary conditions of the type $gV_{(i)}$. To be more specific: we write g as (g_+, g_-), thereby keeping the possibility of left-right asymmetric twists. (The symmetric case is represented by: $g_+ = g_-$.) We deduce with the help of (3.34) that under $\partial_+ X \mapsto g_+ \partial_+ X$, $\partial_- X \mapsto g_- \partial_- X$, the $V_{(i)}$ map as:

$$(g_+, g_-) : \ V_{(i)} \mapsto V_{g(i)} \equiv g_+^{-1} V_{(i)} g_- \tag{3.44}$$

[10]In most compactifications this sector contributes scalars transforming in the adjoint representation as well.

The zero modes map as well. However there is no *a priori* well defined action on the zero modes for asymmetric g. In addition, g as a symmetry admits a unitary (possibly projective) representation on the CP factors: $\Lambda^{(i,j)} \in \mathcal{H}^{\mathrm{L}}_{(i)} \otimes \mathcal{H}^{\mathrm{R}}_{(j)}$ with

$$\Lambda^{(i,j)} \mapsto \left(\gamma_g^{(i)}\right)^\dagger \Lambda^{(g(i),g(j))} \gamma_g^{(j)} \tag{3.45}$$

γ is the matrix representation of g. The G-invariant open-string states have to obey:

$$g\,|\,\psi\,\rangle = |\,\psi\,\rangle \qquad \forall g \in G \tag{3.46}$$

The states in (3.46) are in general linear combinations of states in the ambient space. However g does not mix or exchange left and right end-points. As a result each state will still be built on a CP-Hilbert space of the form $\mathcal{H}^{\mathrm{L}}_{(i,n)} \otimes \mathcal{H}^{\mathrm{R}}_{(j,n)}$. Now the indices i and j need no longer refer to a single pair of boundary conditions $\theta(i,j)$ but they can belong to g-invariant linear combinations. The index n indicates that in general the CP-space depends also on the oscillator excitations. We will not step into a classification of all possible resulting Hilbert-spaces. We only state that the the gauge group of the orbifolded theory is a product

$$\prod_{i=1}^{n} U(n_i) \tag{3.47}$$

of unitary groups $U(n_i)$. The breaking by G occurs in general when the boundary conditions $V_{(j)}$ are g-invariant, and the $U(n_j)$ group is broken down to a product of groups $\prod_i U(n_i)$ by the action of g .

Our analysis was restricted to tree-level. Up to this point it is fully consistent to mod out by twists which act only on the CP space while being trivial in space-time (i.e. not only trivial on the bdy.-condition but trivial on $|\,\psi_{\mathrm{osc}}\,\rangle$, too). However such kind of gauge symmetry breaking would be inconsistent at one-loop order of string perturbation theory.

To see this, we have a look at the non-planar one-loop diagram with four external states in figure 3.6. We assume that the orbifold group G splits (or more appropriately: projects) the CP-Hilbert space such that

$$U(n_1 + n_2) \xrightarrow{\mathcal{P}_g} U(n_1) \times U(n_2) \tag{3.48}$$

Without the projection $\mathcal{P}_g$ there would be (massless) states with CP indices in the off-diagonal blocks, while after the projection, they have vanished (seemingly):

$$\Lambda^{(1\cup 2,1\cup 2)} = \left(\begin{array}{c|c} \Lambda^{(1,1)} & \Lambda^{(1,2)} \\ \hline \Lambda^{(2,1)} & \Lambda^{(2,2)} \end{array}\right) \xrightarrow{\mathcal{P}_g} \Lambda'^{(1\cup 2,1\cup 2)} = \left(\begin{array}{c|c} \Lambda^{(1,1)} & 0 \\ \hline 0 & \Lambda^{(2,2)} \end{array}\right) \tag{3.49}$$

In non-planar loop diagrams the states with values in the off-diagonal components $\Lambda^{(1\cup 2,1\cup 2)}$ re-enter: In figure 3.6 The states $\mathcal{V}_a$ and $\mathcal{V}_b$ transform in the $U(n_1)$ while the two other states $\mathcal{V}_c$ and $\mathcal{V}_d$ transform in the second factor, i.e. the $U(n_2)$. We make this transparent by putting different colors on the two

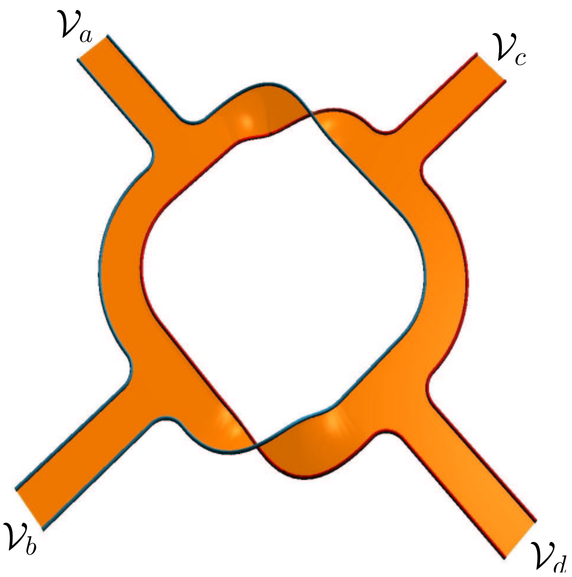


Figure 3.6: Non-planar open-string one-loop diagram with four external states. We assume the states $\mathcal{V}_a$ and $\mathcal{V}_b$ to transform in the $U(n_1)$ while the states $\mathcal{V}_c$ and $\mathcal{V}_d$ transform in the second gauge group, the $U(n_2)$.

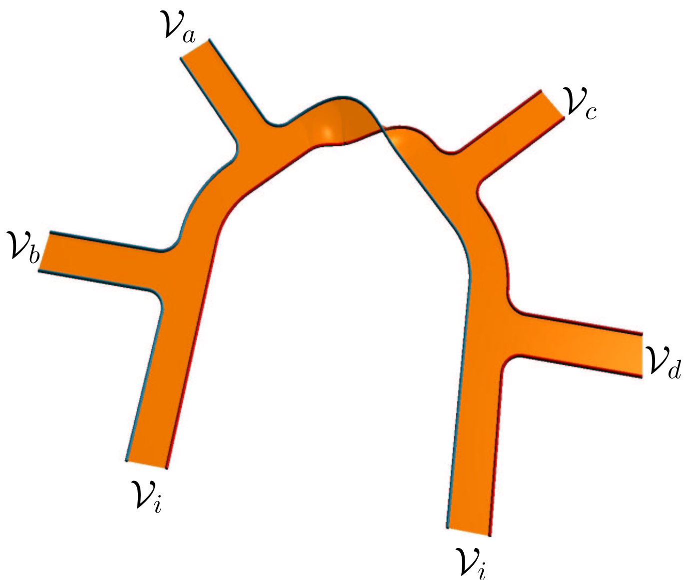


Figure 3.7: After cutting the above non-planar diagram between the external states $\mathcal{V}_b$ and $\mathcal{V}_d$ we obtain the diagram on the left. It has the topology of a disk-diagram.
$\mathcal{V}_i$ corresponds to internal states. $\mathcal{V}_i$ takes Chan-Paton values in the bifundamental of $U(n_1) \times U(n_2)$.

boundaries in the figure. According to the rules, this amplitude is therefore proportional to

$$\mathcal{A}_{a,b,c,d} \propto \operatorname{tr}\big(\Lambda_a^{(1,1)}\Lambda_b^{(1,1)}\big) \cdot \operatorname{tr}\big(\Lambda_c^{(2,2)}\Lambda_d^{(2,2)}\big) \tag{3.50}$$

Unitarity of string theory is manifest in cutting rules. In figure 3.7 we cut the non-planar diagram between the external states (or: strips) $\mathcal{V}_b$ and $\mathcal{V}_d$. The resulting diagram represents a disk-amplitude with six vertex-operator insertions, two of the stemming from the cut, which we denote by $\mathcal{V}_i$. According to our rules, this amplitude is proportional to

$$\mathcal{A}_{a,b,i,c,d,i} \propto \operatorname{tr}\big(\Lambda_a^{(1,1)}\Lambda_b^{(1,1)}\Lambda_i^{(1,2)}\Lambda_c^{(2,2)}\Lambda_d^{(2,2)}\Lambda_i^{(2,1)}\big) \tag{3.51}$$

For unitarity reasons, by summing over internal states $\mathcal{V}_i$ we should obtain the planar amplitude (3.50). This is only possible, if the matrices $\Lambda_i^{(1,2)}$ are non-vanishing for the internal states. As we require unitarity these internal states have to show up in the spectrum. If the space times twist in the orbifold group G are trivial, we get massless vectors in the bifundamental representation of $U(n_1) \times U(n_2)$. These massless vectors can be combined with the massless vectors that transform in the adjoint representation of $U(n_1)$ and $U(n_2)$ to fill the representation adjoint representation of $U(n_1 + n_2)$. This indicates that the full $U(n_1 + n_2)$ gauge symmetry is still present.[11] Thus gauge symmetry breaking by pure gauge twist (i.e. a group G that solely acts on the CP Hilbert space) is forbidden by unitarity. We require a consistent representation of the twists g on both the Chan-Paton wave functions and on the oscillator part $|\psi_{\text{osc}}\rangle$. Other restrictions on the representation stem from tadpole cancellation conditions.

In a similar spirit, we demand the representation of $g \in G$ on the CP-Hilbert space to depend only on the boundary condition $V_{(i)}$ of the corresponding left or right string-endpoint. In other words: it is not possible to split *a priory* the set of D-branes with identical boundary conditions into several sets on which g then acts differently: N_i branes with boundary condition $V_{(i)}$ give always rise to a $U(n_i)$ gauge group which might get broken by a consistently acting orbifold group. However one can modify the boundary conditions by introducing different *Wilson lines* on the respective set of branes, thereby evading the restriction (cf. chapter 7).

There is another subtlety concerning the zero modes appearing in the context of so called *fractional branes* (also: twisted branes). These are branes coming from sectors with $V_{(i)} = V_{g(i)}$: The boundary condition associated with the *single* brane is already symmetric under the group G. Therefore one does not need to introduce its G-pictures. As a consequence they are "smaller" (in the ambient space of the orbifold) than the D-branes which originate from non G-invariant boundary conditions. As branes carry charges, the fractional branes will carry a smaller amount of charge than the objects which are obtained by explicitly symmetrizing over $g \in G$.

So far we have not established a link between the CP-degrees of freedom and the boundary-states. We know however that loop- and tree-channel are only

[11] I want to thank Stephan Stieberger for clarifying the above discussion.

different parameterizations of the same world sheet, and therefore the amplitudes are identical. However closed strings do not carry CP-degrees of freedom. They also do not appear in the definition of the $V_{(i)}$. Keeping in mind the tensor product structure of the CP Hilbert space of the non-orbifolded theory (eq. (3.37)) we notice that the loop-channel trace (in the operator formalism) splits into a product of traces over the CP-Hilbert space times a trace over the boundary conditions. Suppressing the trace over the CP-Hilbert space we get for each CP-sector $\Lambda^{(i,j)}$:[12]

$$\mathrm{A}^{ij}_{ab}(h) = \frac{V_{10}}{(2\pi)^d} \lim_{\epsilon\to 0} \int_\epsilon^\infty \frac{dt}{t} Z_{\mathrm{A}^{ij}_{ab}} \tag{3.52}$$

$$= \int dl \left\langle \mathcal{D}(V_{(i)}); h; a \middle| e^{-2\pi l (H+\widetilde{H})_{h\text{-twisted}}} \middle| \mathcal{D}(V_{(j)}); h; b \right\rangle \tag{3.53}$$

$$Z_{\mathrm{A}^{ij}(h)} = Z_{\mathrm{A}^{ij}_{ab}(h)} \equiv \frac{1}{|O|} \operatorname{tr} h\, e^{-2\pi t (H_{\theta(j,l)})} \tag{3.54}$$

Tracing both expressions over the CP-Hilbert space $\mathcal{H}^{\mathrm{L}}_{(i)} \otimes \mathcal{H}^{\mathrm{R}}_{(j)}$ and taking into account that h acts on it by $\gamma^{(i)}_h \otimes \bar{\gamma}^{(j)}_h$ we can rewrite the complete annulus amplitude in the sector given by $\theta(i,j)$ as:

$$\mathrm{A}^{ij}(h) \equiv \left(\operatorname{tr} \gamma^{(i)}_h\right) \left(\operatorname{tr} \bar{\gamma}^{(j)}_h\right) \frac{V_{10}}{(2\pi)^d} \lim_{\epsilon\to 0} \int_\epsilon^\infty \frac{dt}{t} Z_{\mathrm{A}^{ij}(h)} \tag{3.55}$$

$$= \left(\operatorname{tr} \gamma^{(i)}_h\right) \left(\operatorname{tr} \bar{\gamma}^{(j)}_h\right) \int dl \left\langle \mathcal{D}(V_{(i)}); h; b \middle| e^{-2\pi l (H+\widetilde{H})_{h\text{-twisted}}} \middle| \mathcal{D}(V_{(j)}); h; b \right\rangle \tag{3.56}$$

$$\left| \mathcal{D}(V_{(j)}); h \right\rangle \equiv \operatorname{tr} \bar{\gamma}^{(j)}_h \cdot \left| \mathcal{D}(V_{(j)}); h; b \right\rangle \tag{3.57}$$

Therefore the trace-insertion in loop-channel is mapped to the normalization and to the closed-string twist h of the boundary state. (Note that $\left|\mathcal{D}(V_{(j)}); h; b\right\rangle$ does not depend on the CP-index b due to the assumption that h preserves the structure $\mathcal{H}^{\mathrm{CP}} \otimes \mathcal{H}_{\theta(ij)}$.) The whole annulus contribution can be written as a sum over perfect squares:

$$\mathrm{A} \equiv \sum_{h\in G} \sum_i \sum_j \mathrm{A}^{ij}(h) \tag{3.58}$$

$$= \sum_{h\in G} \int dl\ \langle \mathcal{D}; h |\, e^{-2\pi l (H+\widetilde{H})_{h\text{-twisted}}}\, | \mathcal{D}; h \rangle \tag{3.59}$$

$$|\mathcal{D}; h\rangle \equiv \sum_j \left| \mathcal{D}(V_{(j)}); h \right\rangle \tag{3.60}$$

j runs over a set of G-symmetrized boundary conditions. Taking into account that vacua of different h-twists have vanishing scalar products and observing that the closed-string propagator $\mathcal{P}_{\mathrm{cl}} = \sum_h \int dl\ \exp\bigl(-2\pi l (H+\widetilde{H})_{h\text{-twisted}}\bigr)$

[12] As we have assumed that the $V_{(i)}$ act trivially on the CP-Hilbert space of the ambient space, we assume that this is the case for the orbifold, too. (I.e. the $V_{g(i)}$, $g \in G$ do not carry CP-indices.)

does not mix the twisted sectors, we can rewrite:

$$\mathrm{A} = \int dl \sum_{\substack{i\in G\text{-inv.}\\ \text{set}}} \sum_{\substack{j\in G\text{-inv.}\\ \text{set}}} \big\langle \mathcal{D}(V_{(i)}) \big| e^{-2\pi l(H+\widetilde{H})} \big| \mathcal{D}(V_{(j)}) \big\rangle \tag{3.61}$$

$$\big|\mathcal{D}(V_{(j)})\big\rangle \equiv \mathcal{S}_j \sum_{h\in G} \big|\mathcal{D}(V_{(j)}); h\big\rangle \qquad \text{no sum over } j \tag{3.62}$$

$\mathcal{S}_j$ defines a symmetrization of the boundary conditions in the sector described by $V_{(j)}$. The choice of $\mathcal{S}_j$ leaves some freedom: if a brane is symmetric under $h \in G$ its image may (which would be just a doubling of dofs) or may not be included. Invariance under h includes also the invariance of the zero modes. By the doubling, a non-trivial action γ_h on the CP dofs can be chosen (cf. eq. (3.45)). As the amplitude is proportional to the product of the traces of both γ_h-matrices (cf. eq. (3.55)), the h-twisted part of the corresponding boundary state vanishes, if the corresponding trace $\mathrm{tr}\,\gamma_h$ equals zero. In geometric orbifolds the massless closed-string states of the twisted sectors can be associated with blowing up modes of the singularity. (For Calabi-Yau orbifolds, these fields are contained in $H^{(1,1)}(\mathcal{M})$.) $\mathrm{tr}\,\gamma_h = 0$ would mean that the Dp-brane does not wrap an (exceptional) cycle of the blow-up whereas for $\mathrm{tr}\,\gamma_h \neq 0$ the contrary is true. As we mentioned, without the doubling and $\gamma_h = \mathrm{Id}$ the h-invariant branes (or: boundary states) are called fractional branes. For the D6-branes discussed in chapter 7 this means that fractional branes have to intersect $\mathbb{Z}_2$-fix-points. As they are stuck to fix-points which in turn are associated with certain twisted closed-string sectors, we also refer to these branes as *twisted* branes. Like the cross-cap states, also the boundary states couple to RR and NSNS fields. The coupling to the untwisted (i.e. $h = 0$) closed-string fields for a fractional D-brane is a fraction of what it would be for a brane which was originally not G-invariant in ambient space.

As we have noticed that D-branes have similar couplings to closed-string fields as O-planes, we could check if the addition of D-branes could cancel the tadpole of the Klein bottle. We are especially interested in the cancellation of RR tadpoles as they can not be cured by a Fischler Susskind mechanism. We would also be glad to cancel the NSNS tadpoles as well, because the NSNS tadpoles can also lead to a deformation of the theory to a singular limit. If the model is supersymmetric in both the open- and the closed-string sector, we also have other phenomenologically appealing features like possible solution to the hierarchy problem etc. We will have a closer look at a $\mathbb{Z}_4$-orientifold in chapter 7 which admits supersymmetric solutions.[13] Similar to the Op-planes, the Dp-branes have a low energy effective action, too. The Chern Simons action looks like (cf. [60, 61]):

$$S^{(\mathrm{D}_p)}_{\mathrm{C.S.}} = \mu_p \int_{\mathrm{D}_p} \mathrm{ch}(\mathcal{F}) \sqrt{\frac{\hat{\mathcal{A}}(\mathcal{R}_T)}{\hat{\mathcal{A}}(\mathcal{R}_N)}} \wedge \sum_{q\in\mathrm{RR}} C_q \tag{3.63}$$

[13] Condiditions for a D-brane to be supersymmetric are given in section 4.4 and in section 6.5.1.

μ_p is the Dp-brane charge. ch is the Chern character, $\mathcal{F} = F + B$ the sum of the electro-magnetic $U(1)$ NS-gauge field F and B is the NSNS two-form. $\hat{\mathcal{A}}$ denotes the A-roof (or Dirac) genus:

$$\hat{\mathcal{A}}(\mathcal{R}) = 1 - \frac{p_1(\mathcal{R})}{24} + \frac{7\big(p_1(\mathcal{R})\big)^2 - 4p_2(\mathcal{R})}{5760} + \dots \tag{3.64}$$

Like for the Op-plane, $\mathcal{R}_T$ (and $\mathcal{R}_N$) are the pull-backs of the curvature two-form to the tangent- (and normal-) bundle of the Dp-brane (and p_i are Pontrjagin classes). The Dirac-Born-Infeld action also contains a term that couples to the combination of NS and NSNS fields $\mathcal{F}$:

$$S^{(\mathrm{D}p)}_{\mathrm{DBI}} = T_p \int_{\mathrm{D}q} e^{-\Phi} \sqrt{-\det(G + \mathcal{F})} \tag{3.65}$$

T_p is the D-brane tension. However it is still a field of research how the non-abelian gauge degrees of freedom are correctly incorporated. One method, which is sufficient for all of our tadpole considerations, is simply to trace over the gauge degrees of freedom. Several non-abelian extensions of the above actions have been suggested, motivated by different approaches ([62, 63, 64, 65, 66, 67]).

For consistency we also have to project on $S\Omega$-invariant states in the open-string sector. This leads to non-orientable diagrams with boundaries in the perturbation series. At one-loop level this is the Möbius strip.

3.3.2 $s\Omega$-invariant open-string sector

The $s\Omega$-projection must be also imposed in the open-string sector. We assume that we have already created the orbifold including oriented open-strings as discussed in the last section. We do not expect that the orbifolded theory is fully consistent at this stage. It will in general still suffer from tadpoles. Consistency of the string perturbation expansion will force us to include non-orientable diagrams with boundaries. We will first have a closer look at the spectrum. For $s\Omega$ to be a symmetry, we have to include all $s\Omega$ images of the brane. In the case without $U(1)$-valued electro-magnetic NSNS fields $F^{(i)}$, we only have to make sure that the configuration (i.e. the Hilbert space) is s-invariant. With non-trivial $U(1)$-fields we note that $\Omega : (F_{(i)}, F_{(j)}) \mapsto -(F_{(j)}, F_{(i)})$ (cf. chapter 4).

Since Ω acts as an orientation reversal on $X(\tau, \sigma)$ there are two possibilities:

1. $s\Omega$ interchanges the sector (i) with a different sector $s\Omega(i)$ ((i) describes a G-invariant combination of boundary conditions). In this case the $s\Omega$ projection breaks the gauge group:

$$U(n_i) \times U(n_{s\Omega(i)}) \xrightarrow{\mathcal{P}_{s\Omega}} U(n_i) \tag{3.66}$$

2. The sector (i) is mapped to itself by $s\Omega$. We will consider this case in more detail:

As Ω acts as orientation-reversal, left and right CP-degrees also have to get exchanged in the sector (ii) under the $s\Omega$ action. Ω includes a transposition of

Λ:

$$\left|\Lambda^{(i,i)}\right\rangle \xrightarrow{\Omega} \left|\left(\Lambda^{(i,i)}\right)^T\right\rangle \tag{3.67}$$

(A hermitian conjugation would leave the hermitian matrix $\Lambda^{(i,i)}$ invariant. Hermitian or anti-hermitian matrices do however not form a $\mathbb{C}$-vector space.) $s\Omega$ could also contain an additional $U(n)$-rotation $U_{s\Omega}$. In total we have for $s\Omega$ in this case:

$$s\Omega: \ |\psi(i,i),\mathrm{osc}\rangle \otimes \left|\Lambda^{(i,i)}\right\rangle \mapsto |s\Omega(\psi)(i,i),\mathrm{osc}\rangle \otimes \left|\left(U_{s\Omega}^{(i)}\right)^{-1}\left(\Lambda^{(i,i)}\right)^T U_{s\Omega}^{(i)}\right\rangle \tag{3.68}$$

We note that $\left(U_{s\Omega}^{(i)}\right)^{-1} = \left(U_{s\Omega}^{(i)}\right)^{\dagger} = \left(\overline{U}_{s\Omega}^{(i)}\right)^{T}$. Under a unitary basis-change $V^{(i)}$ in the left or right CP-Hilbert space $\mathcal{H}_{\mathrm{CP}}^{(i)}$ (belonging to the boundary condition (i)), $U_{s\Omega}^{(i)}$ will *not* transform by conjugation:

$$\Lambda^{(i,j)} \mapsto \Lambda'^{(i,j)} = \left(V^{(i)}\right)^{-1}\Lambda^{(i,j)}V^{(j)} \quad \Rightarrow U_{s\Omega}^{(i)} \mapsto U'^{(i)}_{s\Omega} = \left(V^{(i)}\right)^{T}U_{s\Omega}^{(i)}V^{(i)} \tag{3.69}$$

Hence a basis of $\mathcal{H}_{\mathrm{CP}}^{(i)}$ in which $U_{s\Omega}^{(i)}$ is diagonal, does not exist in generic cases. However it turns out that $U_{s\Omega}^{(i)}$ is either symmetric (or anti-symmetric) in which case simple representations exist:
The relation $(s\Omega)^2 = h$, $h \in G$ has to be obeyed at least up to a phase in both the oscillator and the CP-part of the state (and in total without a phase). Because we have already performed the G-projection (*and assuming that the resulting state can be written as a direct product* of a CP-part and a part $|\psi(i,j),\mathrm{osc}\rangle$) this reduces to:[14]

$$(s\Omega)^2\,|\psi(i,j),\mathrm{osc}\rangle = \exp(i\phi_{\mathrm{osc}}^{(i,j)})\,|\psi(i,j),\mathrm{osc}\rangle \tag{3.70}$$

$$\begin{aligned}(s\Omega)^2\left|\Lambda^{(i,j)}\right\rangle =& \exp(i(\phi_{\mathrm{CP}}^{(i)} - \phi_{\mathrm{CP}}^{(j)}) \\ &\cdot\left|\left(((U_{s\Omega}^{(i)})^{-1})^T U_{s\Omega}^{(i)}\right)^{-1}\Lambda^{(i,j)}\left(((U_{s\Omega}^{(i)})^{-1})^T U_{s\Omega}^{(i)}\right)\right\rangle\end{aligned} \tag{3.71}$$

with the phase depending on the super-selection sector (i,j). We have used that the right CP space transforms in the complex conjugate representation with respect to the left one. We also deduce that for identical boundary conditions on left- and right-movers (including the effect of the GSO-projection):

$$\phi_{\mathrm{osc}}^{(i,i)} = 0 \mod 2\pi \tag{3.72}$$

If $(s\Omega)^2$ equals the identity,[15] $U_{s\Omega}^{(i)}$ is either symmetric or anti-symmetric in sectors in which $s\Omega$ leaves the boundary condition invariant:

$$U_{s\Omega}^{(i)} = (U_{s\Omega}^{(i)})^T \qquad \text{or} \qquad U_{s\Omega}^{(i)} = -(U_{s\Omega}^{(i)})^T \qquad \text{for} \quad s\Omega(i) = i \tag{3.73}$$

[14] By this we assume that $|\psi\rangle$ is G-invariant. The split of a general G-invariant state into a product of a CP and an oscillator part is in general not possible. However there should exist a basis of G-invariant states that admits this product structure. Relation (3.72) holds for each of these basis vectors.

[15] In the bosonic string, and on GSO-projected states of the superstring.

By a unitary base change of the form (3.69) we can achieve $U^{(i)}_{s\Omega}$ to be:

$$U^{(i)}_{s\Omega} \qquad \text{symmetric:} \qquad U^{(i)}_{s\Omega} = \mathbb{1}_{n_i} \tag{3.74}$$

$$U^{(i)}_{s\Omega} \qquad \text{anti-symmetric:} \qquad U^{(i)}_{s\Omega} = \left(\begin{array}{c|c} 0 & i\mathbb{1}_{n_i/2} \\ \hline -i\mathbb{1}_{n_i/2} & 0 \end{array}\right) \tag{3.75}$$

This fact is proven in appendix B. The situation is more complicated if $(s\Omega)^2 = h \neq \mathrm{Id}$. To the knowledge of the author, this case is not classified in physics literature. If an element $(s\Omega)^2 = \mathrm{Id}$ is contained in the orientifold-group O, we take this element as the representative in eq. (3.2) (p. 55). This element does not need to be unique, of course. If we know the form of $U^{(i)}_{s\Omega}$ for this element, we may determine the form of the remaining $U^{(i)}_{s\Omega g}$, $g \in G$ by requiring that the U matrices (often called γ-matrices as well, which should not be confused with the generators of the Dirac-algebra) form a representation of the orientifold group O.

Even though all models considered in this thesis belong to the class $(s\Omega)^2 = \mathrm{Id}$, we explain a problem in the case that such an element is not included in O. This case implies that the Klein bottle only leads to twisted-sector tadpoles. The twist corresponds to $(ks\Omega)^2 = h$, with $k, h \in G$. It is not clear, if such kind of twisted tadpoles can be canceled. However it is obvious that the cancellation of purely twisted tadpoles by adding D-branes and leaving the background otherwise unmodified, is impossible. D-branes always couple to untwisted closed-string fields as the partition function can be written in terms of traces (cf. eq. (3.58)). The Id-trace-insertion (which is not allowed to vanish) corresponds to untwisted closed-string exchange in the tree-channel. Therefore each individual annulus amplitude has non-vanishing untwisted closed-string contribution. As the Klein bottle does not contribute to this untwisted closed-string exchange (in the case at hand), the total untwisted annulus tadpole has to vanish by itself. For the RR-tadpole this would imply that both branes and anti-branes (which have the opposite coupling to the RR-fields) are present. As anti-branes have identical boundary conditions as branes, except for the GSO-projection, which is reversed, supersymmetry gets broken. The annulus NSNS tadpole however can not be eliminated since it has the same sign for branes and anti-branes. As a consequence we do not expect that supersymmetric orientifolds exist, with $(hs\Omega)^2 \neq \mathrm{Id}\ \forall h \in G$.

Assuming from *now onwards*, that an *order two element* $s\Omega$ is contained in the orientifold group O, we still have not derived if $s\Omega$ is represented on the CP dofs by a symmetric (3.74) or anti-symmetric matrix (3.75). This is in general not easy to decide. It may get derived from the tadpole-cancellation conditions. However it is often possible to derive relations between different $U^{(i)}_{s\Omega}$ acting on different boundary conditions by the use of the vertex operator algebra. We will sketch one method. We assume that we have a (GSO-invariant) vertex operator $V^{(i,i)}$ that corresponds to the boundary condition (i,i) on which we assume $s\Omega$ to act trivially: $s\Omega(i) = i$. The same we assume for a second boundary condition: $s\Omega(j) = j$. The CP factors are not yet included in the vertex operators. We further assume that we know the explicit form of the vertex operators in the following OPE: [16]

$$\mathcal{V}^{(i,j)}\mathcal{V}^{(j,i)} \sim \mathcal{V}^{(i,i)} \tag{3.76}$$

In addition we require $\mathcal{V}^{(i,i)}$ and $\mathcal{V}^{(j,j)}$ to be $s\Omega$-Eigenstates with known Eigenvalue λ_i resp. λ_j. We are then able to determine the relative sign in the $(s\Omega)^2$ projection on the CP-dofs: Since $s\Omega$ interchanges the left boundary condition i with the right boundary condition j without changing the level (mass) of the vertex operator, we deduce:

$$\mathcal{V}^{(i,j)} = \xi s\Omega\big(\mathcal{V}^{(j,i)}\big), \qquad \xi \in \mathbb{C} \tag{3.77}$$

(The proof relies very much on this fact, i.e. on $s\Omega$ invariant boundary conditions i *and* j.) Now we use that:[17]

$$s\Omega\big(\mathcal{V}^{(i,j)}\mathcal{V}^{(j,i)}\big) = s\Omega\big(\mathcal{V}^{(j,i)}\big)s\Omega\big(\mathcal{V}^{(i,j)}\big) \sim s\Omega\big(\mathcal{V}^{(i,i)}\big) = \lambda_i\mathcal{V}^{(i,i)} \tag{3.78}$$

Inserting relation (3.77) and denoting the $(s\Omega)^2$ Eigenvalue of $\mathcal{V}^{(j,i)}$ by ϵ we get:

$$\epsilon\xi s\Omega\big(\mathcal{V}^{(j,i)}\big)\mathcal{V}^{(j,i)} = \epsilon\mathcal{V}^{(i,j)}\mathcal{V}^{(j,i)} \sim s\Omega\big(\mathcal{V}^{(i,i)}\big) = \lambda_i\mathcal{V}^{(i,i)} \tag{3.79}$$

If (3.76) and $s\Omega(i) = i$ holds, we directly deduce:

$$\left.\begin{aligned} (s\Omega)^2\big(\mathcal{V}^{(i,j)}\big) &= \epsilon\big(\mathcal{V}^{(i,j)}\big) \\ s\Omega\big(\mathcal{V}^{(i,i)}\big) &= \lambda_i\mathcal{V}^{(i,i)} \end{aligned}\right. \Longrightarrow \epsilon = \lambda_i \tag{3.80}$$

Given an $s\Omega$ invariant sector i with $s\Omega$ Eigenvalue $\lambda_i = -1$ for a specific boundary vertex operator $\mathcal{V}^{(i,i)}$ we require that $(s\Omega)^2$ acts as the identity in the (i,j) sector (i.e. on the whole state including the CP dofs). This imposes opposite $(s\Omega)^2$ projections on the CP Hilbert space in the i^{th} and j^{th} sector:

$$U^{(i)}_{s\Omega} = \pm\big(U^{(i)}_{s\Omega}\big) \qquad\qquad U^{(j)}_{s\Omega} = \mp\big(U^{(j)}_{s\Omega}\big) \tag{3.81}$$

(In other words: The action on the CP dofs has to compensate the phase -1 of $(s\Omega)^2$ acting on the oscillators.) This method was used in [68] to derive opposite Ω-projections on D9- and D5-branes. Even though we might have reduced the choices of $U^{(i)}_{s\Omega}$ in this way, we cannot deduce the spectrum directly. First we have to determine the tadpole cancellation conditions, which in addition to the algebraic restrictions further constrain the form of the $U^{(i)}_{s\Omega}$ and $U^{(i)}_{g}$. We will however state the result, that the only gauge groups that can be obtained in the perturbative spectrum of orientifold theories are the $SO(n)$, $USp(n)$ and $U(n)$ groups, as well as direct products of these groups. These restrictions arise

[16] For illustrative reasons, we assume to have operators with this simple OPE. Of course, an asymptotic expansion of the OPE involves in general a sum over vertex operators on the right hand side which are multiplied by different (not necessarily constant) coefficients.

[17] Here we made the assumption that $s\Omega$ exchanges two vertex operators. In principle it could also exchange the vertex operators and multiply the resulting product by -1. This second possibility would reverse the conclusions in such a way that both matrices $U^{(i)}_{s\Omega}$ and $U^{(j)}_{s\Omega}$ would have the same symmetry properties in (3.81).

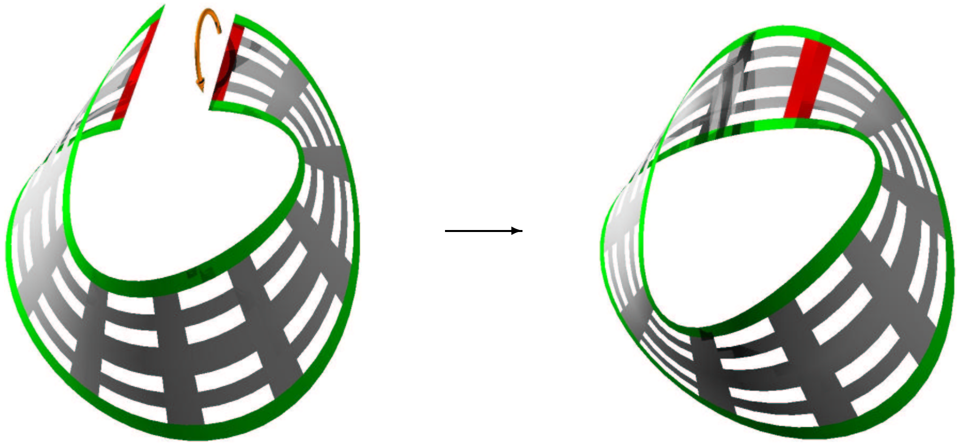


Figure 3.8: Construction of the Möbius strip by gluing two ends of a twisted strip in the way depicted.

if one imposes factorization of open-string amplitudes (which is needed in order that the theory is consistent with unitarity) [69, 70]. No simple rule is known to deduce the spectrum of generic orientifolds directly (except for some classes of orientifolds like the models considered in [43]). For the models presented in this thesis, we will find consistent actions of the orientifold group O on the CP Hilbert space, which allow a projection of the open-string spectrum onto an O-invariant subspace.

3.3.3 Möbius amplitude

The remaining $\chi = 0$ diagram is the Möbius strip. Topologically, it is obtained form a strip, with the two ends twisted and then glued together, such that the resulting object is non-orientable (cf. figure 3.8). This picture corresponds to the loop-channel in which an open-string circulates in a loop. Like the Klein bottle, the Möbius strip is obtained by moding out another world sheet by a $\mathbb{Z}_2$-involution. For the Möbius strip this ambient world sheet is the annulus. Like for the Klein bottle we paint a diagram, from which we will read off the periodicities (figure (3.9)).

The involution for the Möbius strip is the same as for the Klein bottle (cf. (3.4), p. 55). Similarly the relations (3.8) and (3.9) (p. 56) between the trace-insertion and the twist in the tree-channel are valid for the Möbius strip as well. In tree-channel only the cross-cap condition (3.12) (p. 58) is valid, if we take σ to be half the length of the open and *not* of the closed-string. The periodicity w.r.t. the $\mathbb{Z}_2$-involution relates both boundary conditions $V_{(1)}$ and $V_{(2)}$, if we

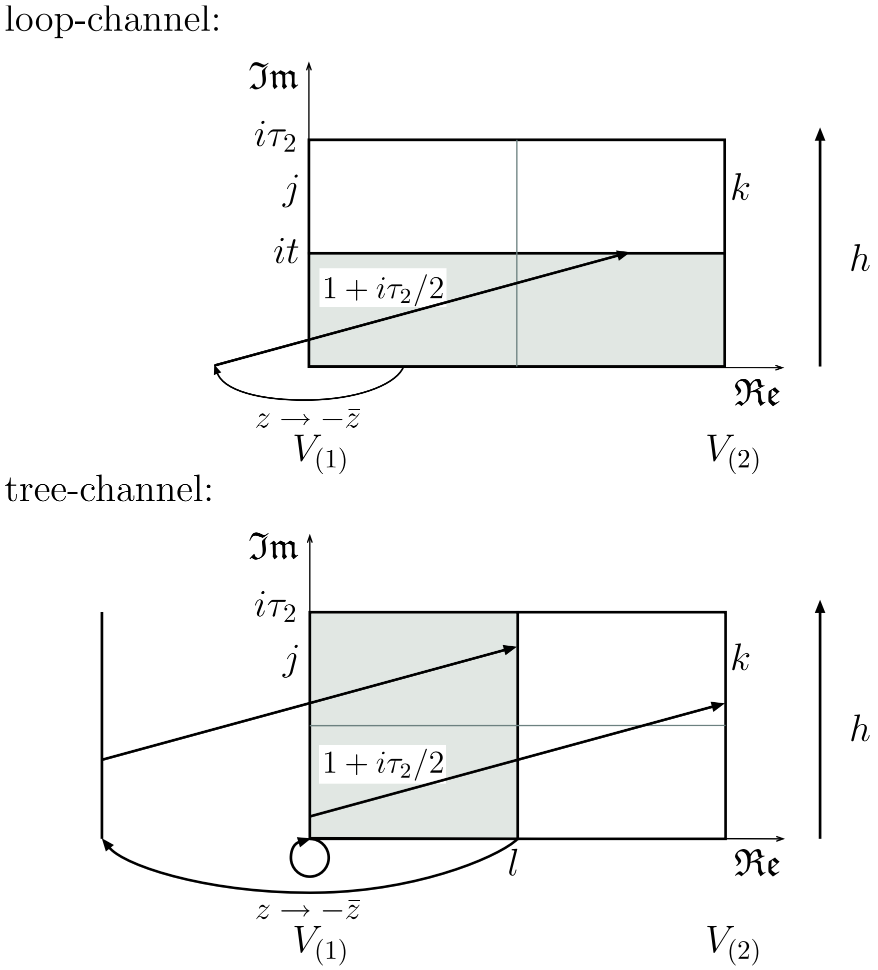


Figure 3.9: Periodicities of the Möbius strip embedded in the underlying annulus and the two fundamental regions (shaded areas). The boundaries correspond to boundary conditions $V_{(1)}$ and $V_{(2)}$. Chan Paton labels j and k are also included in the diagrams.

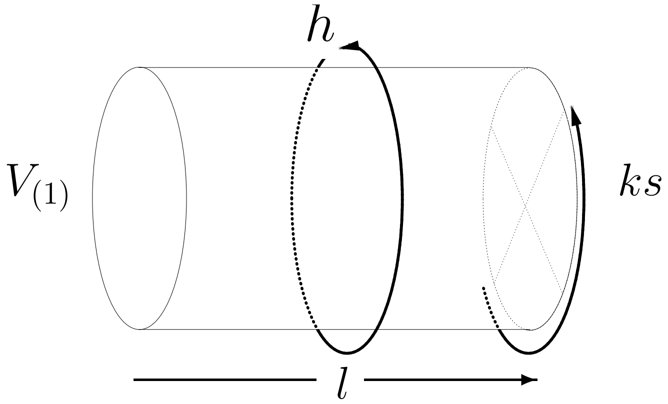


Figure 3.10: The Möbius strip in the tree-channel. It describes a closed-string that is emitted from a D-brane described by boundary condition $V_{(1)}$ and absorbed by the cross-cap state (or O-plane) $|\mathcal{C}(ks)\rangle$ after world sheet time l.

have a trace-insertion $ks\Omega$ in the corresponding partition function:

$$
\begin{aligned}
\partial_\mp X(\tau,\sigma=0) &= ks_\pm \partial_\pm X(\tau+t,\sigma=1) && (3.82)\\
&= ks_\pm V_{(2)}^{\pm 1}\partial_\mp X(\tau+t,\sigma=1) && (3.83)\\
&= ks_\pm V_{(2)}^{\pm 1} ks_\pm \partial_\pm X(\tau+2t,\sigma=0) && (3.84)\\
ks_\pm ks_\mp \partial_\mp X(\tau+2t,\sigma=0) &= ks_\pm V_{(2)}^{\pm 1} ks_\pm V_{(1)}^{\mp 1}\partial_\mp X(\tau+2t,\sigma=0) && (3.85)
\end{aligned}
$$

Thus we found the necessary condition for the boundary conditions of the Möbius strip:[18]

$$ks_- V_{(1)} = V_{(2)} ks_+ \tag{3.86}$$

Similar conditions hold for the zero modes as well. In addition the CP states $\Lambda^{(i,j)}_{a;b}$ have to obey:

$$\Lambda^{(i,j)}_{ab} = \left(\left(U^{(j)}_{s\Omega}U^{(j)}_{k}\right)^{-1}\left(\Lambda^{(i,j)}\right)^T U^{(i)}_{s\Omega}U^{(i)}_{k}\right)_{ab} \tag{3.87}$$

The tree-channel of the Möbius-amplitude is depicted in figure 3.10. The relation between loop- and tree-channel parameter t and l is listed in table 3.2 (p. 64). It is obtained by the same reasoning as for the Klein bottle, except that the open-string length is half of the closed-string length. Like the other diagrams, we write the Möbius amplitude in loop-channel and in tree-channel, where it corresponds to a closed-string exchange between a boundary and a

[18]The index $\pm$ on ks takes into account that ks might act differently on left- and right-movers in the case of asymmetric orientifolds.

cross-cap state.

$$\mathrm{M}^i(ks) = \mathrm{tr}\Big(\bar{\gamma}^{(i)}_{ks\Omega}\big(\gamma^{(i)}_{ks}\big)^{-1}\gamma^{(i)}_{ks\Omega}\big(\gamma^{(i)}_{ks}\big)\Big)\cdot\frac{V_{10}}{(2\pi)^d}\lim_{\epsilon\to 0}\int_\epsilon^\infty\frac{dt}{t}Z_{\mathrm{M}^i} \tag{3.88}$$

$$= \int dl\,\big\langle\mathcal{C}(ks)\big|e^{-2\pi l(H+\widetilde{H})_{h=(k\Omega s)^2\text{-twisted}}}\big|\mathcal{D}(V_{(i)});h\big\rangle + \text{c.c.} \tag{3.89}$$

$$Z_{\mathrm{M}^i(h)} = Z_{\mathrm{M}^i_{ab}(ks\Omega)} \equiv \frac{1}{|O|}\underset{\substack{(V_i,(ks)^{-1}V_i(ks))\\ \setminus\mathcal{H}_{\mathrm{CP}}}}{\mathrm{tr}}\,ks\Omega\,e^{-2\pi t(H_{\theta(jl)})} \tag{3.90}$$

The subscript under the tr indicates that the CP-trace is excluded in this expression in accordance with eq. (3.55). Condition (3.86) is already imposed. We have used the common notation $U_g^{(i)} = \gamma_g^{(i)}$ for the representation of O on the CP Hilbert space. Summing $\mathrm{M}^i(ks)$ over all $i \in \{\text{bdy-conditions}\}$ as well as over all trace-insertions $ks\Omega$ compatible with (3.86), we obtain the whole Möbius amplitude. Written in tree-channel it takes the form:

$$\mathrm{M} = \int dl\sum_{h\in G}\big\langle\mathcal{C}|s;h\big|e^{-2\pi l(H+\widetilde{H})_{h^2\text{-twisted}}}\big|\mathcal{D};h\big\rangle + \text{c.c.} \tag{3.91}$$

The fact that eq. (3.89) and (3.91) take this special form is highly non-trivial (and we do not prove it). It follows from the normalization of the cross-cap and boundary states, which is determined by rewriting the Klein bottle and annulus loop-amplitudes as tree-channel amplitudes. This form (especially the prefactor) is deeply linked to the dimension of the target space. For general CFTs it can be different. However in all models presented in this paper, this factor is present. It is important for rewriting the whole amplitude as a perfect square:

$$\begin{aligned}&\mathrm{KB}+\mathrm{M}+\mathrm{A}\\ &= \int dl\sum_{h\in G}\big(\langle\mathcal{C}|s;h| + \langle\mathcal{D};h|\big)e^{-2\pi l(H+\widetilde{H})_{h^2\text{-twisted}}}\big(|\mathcal{C}|s;h\rangle + |\mathcal{D};h\rangle\big)\end{aligned} \tag{3.92}$$

In addition, this factorization imposes conditions on the representation matrices $\gamma^{(i)}_{\dots}$. The ability to write this amplitude as a perfect square leads to its factorization in the limit of huge world-sheet-times l. The cancellation of the overall tadpole requires the $l\to\infty$ limit of the integrand in (3.92) to vanish separately for each physically distinguishable closed-string excitation. In this limit only the closed-string q^0-term, i.e. massless modes can contribute. Each independent tadpole must be canceled separately. In the $\mathbb{Z}_4$-orientifold of chapter 7 this means that branes wrapped around blown up fix-points have to cancel their twisted sector charges on each fix-point individually. Furthermore, physically different tadpoles might be distinguished by their dependence on geometrical data like the volume (complex structure, etc.). The tadpole cancellation conditions impose constraints both on the allowed boundary conditions $V_{(i)}$ and on the representation matrices γ of the orientifold group O. These constraints still leave some freedom in many cases. Requiring supersymmetry can further reduce this freedom and leads in some cases to a unique solution. To be more specific: The Id trace-insertion determines the dimension(s) of the CP-Hilbert space(s). This is usually identified with the number of D-branes (though different authors sometimes include or exclude the $s\Omega$ and/or G images of the branes in their counting). The $s\Omega$-insertion (i.e. the Möbius amplitude without further insertions) also fixes the form of the $U^{(i)}_{s\Omega}$ matrices. Usually the Id sector leads to a binomial formula of the type $(N-C)^2 \overset{!}{=} 0$. C is associated with the coupling of the cross-cap to the field that generates the tadpole. In the simplest case N counts the number of branes. More generally it includes topological data for the RR sector and data which depends on differential (i.e. not purely topological) properties of the brane for NSNS sector tadpoles. If different untwisted sectors are present one gets several of these binomial formulæ. Non trivial trace-insertions lead generically to twisted sector tadpoles. They constrain the form of the $U^{(i)}_g$ (or $\gamma^{(i)}_g$) matrices. Often a geometrical interpretation of some invariants like $\operatorname{tr} U^{(i)}_g$ is accessible. In the $\mathbb{Z}_4$ models of chapter 7 this trace is interpreted as the wrapping number of the brane (i) around (blown-up) $g \in \mathbb{Z}_2$-twisted cycles. As boundary states have often an interpretation in terms of geometrical D-branes, which are (connected) sub-manifolds, one could also determine the tadpole conditions via the low-energy effective actions like (3.31) (p. 63) and (3.63) (p. 73). However one should have in mind, that an interpretation in terms of partition functions always has to exist in order to give a sensible string interpretation.

Instead of modifying the open-string background, one could also try to modify the closed string background, thereby solving the eoms [71]. For example the NSNS 3-form field strength couples naturally to the RR 4-form potential as well as to the RR 3-form field strength in the CS-action of Type IIB theory.[19] One could also combine both possibilities. However the stringy description (in form of a CFT solution) of a non-trivially modified closed-string background is often not known. This is also true for most non-linear, i.e. X-dependent boundary conditions $V_{(i)}$, that relate the left- and right-moving parts of the open-string in addition to specifying its zero-modes. However linear boundary conditions are solved more easily. By this one obtains usually a great variety of different, and often phenomenological appealing solutions including chiral fermions transforming in interesting gauge-groups.[20]

3.3.4 Orientifolds of supersymmetric strings

In this section we will make some comments about orientifolds of supersymmetric string theories. In addition to the bosonic string sector, world-sheet fermions appear. The fermionic term of the gauge fixed fermionic action looks like:

$$S_{\text{ferm}} = -\frac{1}{4\pi\alpha'} \int_{\mathcal{M}} d^2\sigma\, 2i\, \bar{\psi}^{\mu} \rho^{\alpha} \partial_{\alpha} \psi_{\mu} \tag{3.93}$$

[19] This fact was used in [72] to construct orientifolds with background fluxes

[20] However exceptional groups are not contained in perturbative orientifold spectra.

The ψ´s are world-sheet majorana spinors. The two dimensional Dirac matrices ρ^α are in this gauge ($h = \begin{pmatrix} h_{\tau\tau} & h_{\tau\sigma} \\ h_{\sigma\tau} & h_{\sigma\sigma} \end{pmatrix} = \mathrm{diag}(-1,1)$) :

$$\rho^0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad \rho^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{3.94}$$

For open-strings it is well known, that one has to include altered boundary conditions (cf. (3.34)):

$$\psi_+(\tau,\sigma) = \kappa_i V_{(i)} \psi_-(\tau,\sigma)\,,\ \sigma \in \partial\mathcal{M}_i \tag{3.95}$$

The relative sign $\kappa_i\kappa_j = \pm 1$ of a string in the (i,j) sector determines whether the open-string belongs to the Ramond (+) or Neveu Schwarz sector (−). As we have to fix the κ_i, this choice is obviously asymmetric in the Neveu Schwarz sector. In the following we write a spinor as:

$$\psi(\tau,\sigma) = \begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix}(\tau,\sigma) \tag{3.96}$$

We have not yet determined how the $\mathbb{Z}_2$-involution $s\Omega$ (cf. eq. (3.4) , p. 55) acts on the world sheet fermions. The trace includes a GSO projection as well. For open-strings:

$$\mathcal{P}_{\mathrm{GSO}} = \frac{1+(-1)^f}{2} \tag{3.97}$$

with f the world sheet fermion number and for closed-strings:

$$\mathcal{P}_{\mathrm{GSO}} = \frac{1+(-1)^{f_{\mathrm{L}}}}{2} \cdot \frac{1+(-1)^{f_{\mathrm{R}}}}{2} \tag{3.98}$$

In order to have a sensible mapping between path-integral and operator formalism we have to be able to describe the action of $s\Omega$ on the fermionic fields in such a way, that they are compatible with allowed boundary conditions in the path-integral. We will consider the three amplitudes seperately:

3.3.4.1 Fermionic sector of the Klein bottle

In the ferionic sector, additional signs can be inserted in the boundary conditions compared to the bosonic sector. For example a fermion is usually antiperiodic in τ-direction. The GSO projection adds however the time-periodic boundary condition as well. In operator formalism, this corresponds to a $(-1)^f$ insertion, f being the world sheet fermion number (possibly restricted to the left- or right-moving sector). A Ramond fermion is periodic in σ, while a Neveu-Schwarz fermion is antiperiodic in this direction. This describes the situation for the torus amplitude. For the Klein bottle and the two remaining $\chi = 0$ amplitudes, we will proceed in the spirit of [73]: We will first add all possible signs in to the bosonic boundary conditions and determine further restrictions on these signs afterwards. For the following discussion we refer again to figure 3.1 (p. 57). There are two possible signs in the τ-direction:

$$h\psi(\tau,\sigma) = \mathrm{diag}(\epsilon_1,\epsilon_2)\,\psi(\tau+2t,\sigma) \tag{3.99}$$

Furthermore, we know that $ks\Omega$ exchanges left- and right-movers. In the fermionic sector there are possibly additional signs. We will write the fermionic analog of $ks\Omega$ as

$$ks\Omega = ks \begin{pmatrix} 0 & \chi_1 \\ \chi_2 & 0 \end{pmatrix} \tag{3.100}$$

Furthermore there is a twist that determines if the string is in the NS or R (precisely NSNS, RR, NSR or RNS) sector of the loop-channel:

$$g\psi(\tau,\sigma) = \mathrm{diag}(\kappa_1,\kappa_2)\,\psi(\tau,\sigma+1) \tag{3.101}$$

From condition (3.8) we derive:

$$(ks\Omega)^2 = h \qquad\qquad \Longrightarrow \chi_1\chi_2 = \epsilon_1 = \epsilon_2 \tag{3.102}$$

$$(gks\Omega)^2 = h \qquad\qquad \Longrightarrow \chi_1\chi_2\kappa_1\kappa_2 = \epsilon_1 \tag{3.103}$$

With (3.102) we get from (3.103):

$$\kappa_1 = \kappa_2 \tag{3.104}$$

The above formula states that in the Klein bottle fields have to be either of RR- or NSNS-type in the loop-channel. The same is also true for the tree-channel due to (3.102). One sign in (3.100), e.g. χ_2 can be eliminated by a field redefinition. The fermionic part of the two crosscaps is now determined by the conditions:

$$\widetilde{\mathrm{KB}}_{(ks,ksg)} = \int_{\mathbb{R}^+} dl\,\big\langle \mathcal{C}(ks)\big| e^{-2\pi l(H+\widetilde{H})_{h\text{-twisted}}} \big|\mathcal{C}(ksg)\big\rangle \text{ with } h=(ks\Omega)^2 \tag{3.105}$$

$$\begin{aligned} \big(\psi_-(\tau=0,\sigma) - ks\psi_+(\tau=0,\sigma+1/2)\big)_{h\epsilon\text{-twisted}}\,|\mathcal{C}(ks)\rangle &= 0 \\ \big(\psi_+(\tau=0,\sigma) - \chi ks\psi_-(\tau=0,\sigma+1/2)\big)_{h\epsilon\text{-twisted}}\,|\mathcal{C}(ks)\rangle &= 0 \end{aligned} \tag{3.106}$$

Similarly we get at the other end:

$$\begin{aligned} \big(\psi_-(\tau=l,\sigma) - \kappa gks\psi_+(\tau=l,\sigma+1/2)\big)_{h\epsilon\text{-twisted}}\,|\mathcal{C}(ks)\rangle &= 0 \\ \big(\psi_+(\tau=l,\sigma) - (\chi\kappa)gks\psi_-(\tau=l,\sigma+1/2)\big)_{h\epsilon\text{-twisted}}\,|\mathcal{C}(gks)\rangle &= 0 \end{aligned} \tag{3.107}$$

The interpretation is as follows: χ determines the GSO projection in the loop-channel. As $\chi = \epsilon$ its sign determines whether a state belongs to the RR or NSNS sector in tree-channel. The sign of κ determines whether a state belongs to the NSNS or RR sector in loop-channel. In tree-channel it shows up in an additional sign in the l-direction. This sign is the analog of a trace-insertion in the torus amplitude. We will therefore distinguish the tree-channel sector by a sign as well. In table 3.3, where all tree-loop channel relations are listed, κ is denoted by $+1$ or -1. A general crosscap state is now denoted as follows:

$$\begin{aligned} \big(\psi_-(\tau=0,\sigma) - \kappa a\psi_+(\tau=0,\sigma+1/2)\big)_{a^2\epsilon\text{-twisted}}\,|\mathcal{C}(a)|\chi,\kappa\rangle &= 0 \\ \big(\psi_+(\tau=0,\sigma) - (\chi\kappa)a\psi_-(\tau=0,\sigma+1/2)\big)_{a^2\epsilon\text{-twisted}}\,|\mathcal{C}(a)|\chi,\kappa\rangle &= 0 \end{aligned} \tag{3.108}$$

$\chi = +1$ is the NSNS, $\chi = -1$ the RR sector. The bosonic part fulfills the same conditions as in (3.15) (p. 59). The complete boundary state is a sum over all four possible choices for χ, κ.

χ	κ	loop-channel	tree-channel
-1	-1	(NSNS,1)	(NSNS,+)
1	-1	(NSNS,$(-1)^f$)	(RR,+)
-1	1	(RR, 1)	(NSNS,$-$)
1	1	(RR, $(-1)^f$)	(RR,$-$)

Table 3.3: Klein bottle, Cylinder: Relation between the fermionic sectors in tree- and loop-channel. For the cylinders´ loop-channel NSNS and RR mean the NS- and R-sector respectively.

χ	κ	loop-channel	tree-channel
$+1$	-1	(NS,1)	(NS,$-$)
-1	-1	(NS,$(-1)^f$)	(NS,+)
1	1	(R, 1)	(R,$-$)
-1	1	(R, $(-1)^f$)	(R,$-$)

Table 3.4: Möbius strip: Relation between fermionic sectors in tree- and loop-channel.

3.3.4.2 Fermionic sector of the Cylinder

The same program can be applied to the cylinder as well. Equation (3.99) is valid (with the signs free), but there is no condition from $ks\Omega$. However we get two conditions from (3.95). From (3.99) and (3.95) we get like for the Klein bottle:

$$\epsilon \equiv \epsilon_1 = \epsilon_2 \tag{3.109}$$

However both signs κ_1 and κ_2 in (3.95) from both boundaries are free. One sign, e.g. κ_2 can be fixed to one by a field redefinition s.th. $\kappa \equiv \kappa_1$ is the second free parameter. The interpretation of these signs is the same as for the Klein bottle. However in loop-channel the NSNS sector is the NS sector for the cylinder. The same is valid for the RR sector. Therefore the tree-loop channel relations can be read off from table 3.3 as well. The fermionic boundary condition in one fermionic sector specified by ϵ and κ reads:

$$\big(\psi_+(\tau=0,\sigma) - \kappa V \psi_-(\tau=0,\sigma)\big)_{\epsilon h\text{-twisted}} |\mathcal{D}(V); h|\epsilon,\kappa\rangle = 0 \tag{3.110}$$

For the bosons the corresponding boundary condition is given by (3.35) (p. 66).

3.3.4.3 Fermionic sector of the Möbius strip

For the Möbius strip the cylinder relations are inherited. However the $ks\Omega$ action imposes additional conditions. The fermionic action of $ks\Omega$ is again given by (3.100). We can apply (3.102) for the Möbius strip, too. We are lead to the same result: $\chi_1\chi_2 = \epsilon$. The Möbius strip condition (3.86) (p. 80) that relates the boundary conditions V_1 and V_2, leads in the fermionic sector to:

$$\chi_1\chi_2 = \kappa_1\kappa_2 \quad (= \epsilon) \tag{3.111}$$

Field redefinition can fix one sign of $\{\chi_1, \chi_2, \kappa_1, \kappa_2\}$. We fix $\kappa_2 = 1$ leaving $\kappa \equiv \kappa_1$ as a free parameter. κ determines whether the fermion belongs to NS- ($\kappa = 1$) or to R-sector ($\kappa = -1$) in loop-channel. In contrast to Klein bottle and cylinder, the NS-sector in loop-channel corresponds to the NSNS sector in tree-channel (c.f. (3.111)). There is still the freedom to choose an overall sign in $ks\Omega$. We define $\chi \equiv \chi_1$ (with $\chi_2 = \kappa/\chi$ imposed by (3.111)). The relations between different sectors in tree- and loop-channel are given in table 3.4.

Concluding remarks

We presented basic notions of orientifolds, including subjects like tadpoles, cross-cap- and more generally: boundary-states, Chan Paton factors and non-orientable world-sheets. Special emphasis was put on the open-closed string correspondence of the one-loop vacuum amplitudes. Of course, we could not cover all subjects. Boundary states for examples can be defined via more general symmetries than the ones derived from the chiral fields $\partial_\pm X^\mu$ (cf. [74,75]). We also excluded the ghost sector from our discussion, as we work in light-cone quantization in most of the following chapters. We have not been very precise in specifying the boundary states, i.e. solving the boundary state conditions. This will however be done in some examples in the following chapters.

We also restricted to the case of orientifolds that stem from orbifolds of smooth manifolds. We want to mention that other interesting constructions exist including for example orientifolds on WZNW-models (describing smooth manifolds as well) and Coset-spaces (cf. [76,77,78,79]). The spectrum of strings and D-branes can be investigated by advanced means of the corresponding CFT. Also more geometrical approaches to D-branes and open strings have been undertaken (cf. [31] and references given there). The relatively simple orbifolds (though interesting as they usually contain singularities) serve as valuable examples and give hints to more general (mathematical) descriptions of D-branes.

There exists a huge amount of literature on orientifolds and it is impossible to cite all of the publications. Some articles with major contributions to the field have been cited in the text, many others not. In the remaining chapters we will mention additional publications, many dealing directly with orientifolds. Not many survey articles have been written on orientifolds however, so it is not too hard to mention some. Polchinski, who did many important work, introduces orientifolds in his two books on string theory [7, 8]. Even though the name "orientifold" was not in use around that time, some basics about open (oriented and non-oriented) superstring theories can be found in the two volumes written by Green, Schwarz and Witten [4, 5]. In 1997, Atish Dabholkar gave a lecture on orientifolds in Trieste. The notes can be found in the internet [80]. More recently Angelantonj and Sagnotti published an overview article [18] that is devoted to the CFT-oriented approach to orientifolds, developed by Sagnotti and collaborators from the mid-eighties onwards.

Even though the material is not complete, we hope that it gives the reader necessary tools to follow the sucsessive text.

Chapter 4

Open Strings in Electro-Magnetic Background-Fields

In this chapter we will quantize the bosonic open string with linear boundary conditions in flat space-time. These are given both by the fact that D-branes are lower dimensional hypersurfaces the string end-points are confined to and by the constant NSNS two-form B in combination with the NS $U(1)$-field strength F. The generalization to superstrings is straightforward and simpler than the bosonic case.[1] We will generalize the result of [83,84] on the non-commutativity of string fields $X^\mu(\tau, \sigma)$ located at the boundary to the one loop case (in comparison with [83]) and to the case where the boundary conditions on both boundaries are given by NS field-strengths F_1 and F_2 that are constant, but completely independent form each other (in comparison with [84]). There are many other approaches to derive the commutator as well. A prominent one is by deformation quantization [85], others are guided by constrained (or Dirac) quantization [86, 87]. Laidlaw calculated the propagator for the cylinder with independent, constant $U(1)$ F-fields at the boundaries and reproduced the result for the commutator as well [88]. The list is surely not complete. We want to mention that the approach to solve the string boundary conditions, which is actually a variant of a doubling trick, was motivated by [89, 90, 91] where the boundary condition problem for open strings on intersecting D-branes of arbitrary dimension *without* NS and NSNS background fields was solved.[2] However we adopted another quantization method here: We first calculate the canonical two-form in terms of the individual modes. Then we restrict to its invertible part. The inverse is (up to a factor) the Poisson-bracket.

We present a new and (to our knowledge) first direct derivation of the open

[1]There is a subtlety in deriving the boundary conditions for the world sheet fermions from the action. Instead of coupling the fermions to the boundary-$U(1)$-field strength F via a boundary term in the action, which might over-constrain the problem, we impose fermionic boundary conditions *by hand*. We demand these fermionic boundary conditions to be compatible with super-symmetry transformations which also have to be reduced (to one half of the bulk super-symmetry) at the boundary (i.e. super-symmetry invariance under the full Majorana spinor ϵ (cf. eq. (4.7)) over-constrains the problem, too). See also [81, 82].

[2]Of course the term *D-brane* was not used around that time. The authors also gave no space-time interpretation of the boundary conditions in the spirit of [92].

string mass formula for toroidally compactified D-branes with magnetic fields (section 4.2.1.3).

4.1 Action and boundary conditions of the open string

We consider the following superconformal-gauge action (space-time metric $G_{\mu\nu}$ is of signature $(1, d-1)$):

$$S = S_{\text{bos}} + S_{\text{ferm}} \tag{4.1}$$

with the convention $\epsilon^{\tau\sigma} = 1$:

$$\begin{aligned} S_{\text{bos}} &= S_{\text{bos}}^{\text{bulk}} + S_{\text{bos}}^{\text{bdy}} \\ &= -\frac{1}{4\pi\alpha'}\left(\int_{\mathcal{M}} d^2\sigma\, \partial_\alpha X^\mu \partial^\alpha X_\mu - B_{\mu\nu}\epsilon^{\alpha\beta}\partial_\alpha X^\mu \partial_\beta X^\nu + 2\int_{\partial\mathcal{M}} d\tau\, \dot{X}^\mu A_\mu\right) \end{aligned} \tag{4.2}$$

$$S_{\text{ferm}} = -\frac{1}{4\pi\alpha'}\int_{\mathcal{M}} d^2\sigma\, 2i\,\bar{\psi}^\mu \rho^\alpha \partial_\alpha \psi_\mu \tag{4.3}$$

In case of a constant $U(1)$-field strength with the gauge $A_\nu = \frac{X^\mu}{2}F_{\mu\nu}$, ($F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$) and constant $B_{\mu\nu}$ equation (4.2) reduces to:[3]

$$S_{\text{bos}} = -\frac{1}{4\pi\alpha'}\left(\int_{\mathcal{M}} d^2\sigma\, \partial_\alpha X^\mu \partial^\alpha X_\mu - 2\dot{X}^\mu (B)_{\mu\nu} X'^\nu - \int_{\partial\mathcal{M}} d\tau\, \dot{X}^\mu (F)_{\mu\nu} X^\nu\right) \tag{4.4}$$

The ψ's are world-sheet Majorana spinors. The two dimensional Dirac matrices ρ^α are in this gauge ($h = \left(\begin{smallmatrix} h_{\tau\tau} & h_{\tau\sigma} \\ h_{\sigma\tau} & h_{\sigma\sigma}\end{smallmatrix}\right) = \text{diag}(-1,1)$) :

$$\rho^0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},\ \rho^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{4.5}$$

They satisfy the algebra:

$$\{\rho^\alpha, \rho^\beta\} = 2h^{\alpha\beta} \tag{4.6}$$

The spinor conjugate to $\lambda := \left(\begin{smallmatrix} \lambda_+ \\ \lambda_- \end{smallmatrix}\right)$ is $\bar{\lambda} := \lambda^\dagger \rho^0 = (-\lambda_-, \lambda_+)$. The charge conjugation matrix C is defined as ρ^0. Then a Majorana spinor is real. The action 4.1 is invariant under the following bulk super-symmetry transformation:

$$\delta_\epsilon X^\mu = i\bar{\epsilon}\psi^\mu \qquad \delta_\epsilon \psi^\mu = \frac{1}{2}\rho^\alpha\left(\partial_\alpha X^\mu\right)\epsilon \tag{4.7}$$

In the case of constant $G_{\mu\nu}$ the variation of the bosonic action with respect to X^μ gives the bulk equation of motion

$$\left(\partial_\tau^2 - \partial_\sigma^2\right) X^\mu = 4\partial_+\partial_- X^\mu = 0 \tag{4.8}$$

[3] Then $B_{\mu\nu}\epsilon^{\alpha\beta}\partial_\alpha X^\mu \partial_\beta X^\nu = B_{\mu\nu} d(X^\mu \cdot dX^\nu) = d(B_{\mu\nu}(X^\mu \cdot dX^\nu))$ with d the exterior derivative on the world sheet.

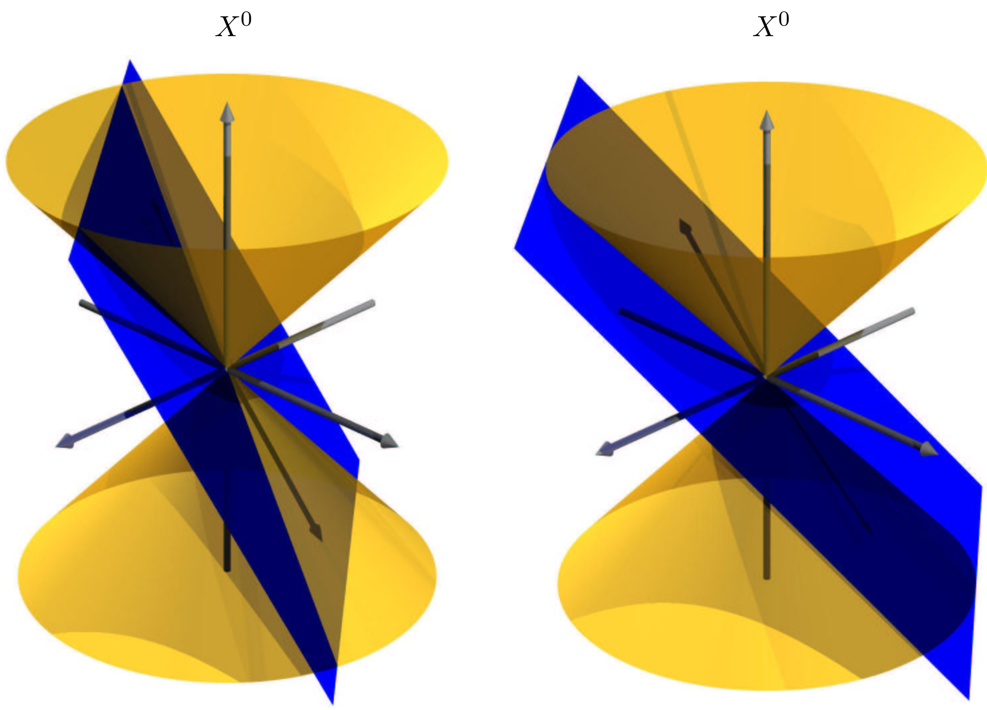


Figure 4.1: Time-like and light-like branes: D-branes can intersect the light-cone (left) or only touch it (right) . We will not consider the case of space-like branes.

with $\partial_\pm \equiv \frac{1}{2}(\partial_\tau \pm \partial_\sigma)$ plus a boundary condition. The boundary contribution to δS_{bos} is:

$$\begin{aligned}
\delta S_{\text{bos, bdy.}} = \\
&-\frac{1}{2\pi\alpha'}\int_{\partial\mathcal{M}} d\tau\, \delta X^\mu \cdot \left(\partial_\sigma X_\mu + B_{\mu\nu}\partial_\tau X^\nu + \partial_\mu A_\nu \partial_\tau X^\nu - \partial_\nu A_\mu \partial_\tau X^\mu\right) \\
= &-\frac{1}{2\pi\alpha'}\int_{\partial\mathcal{M}} d\tau\, \delta X^\mu \cdot \left(\partial_\sigma X_\mu + \mathcal{F}_{\mu\nu}\partial_\tau X^\nu\right) \text{ with } \mathcal{F}_{\mu\nu} \equiv (B+F)_{\mu\nu} \qquad (4.9)
\end{aligned}$$

We will consider *flat* D-branes of arbitrary dimension with *constant* but otherwise completely general B and $U(1)$ background flux F. Then these D-branes are hyperplanes. The $\mathbb{R}^d$ can be decomposed as $D_p \oplus V_{d-(p+1)}$ with $V_{d-(p+1)}$ the orthogonal compliment of the Dp-brane D_p. $\mathcal{P}_\parallel$ and $\mathcal{P}_\perp$ denote the parallel resp. tranverse projections with respect to the brane. They can be defined as follows: Let the D-brane be spanned by a set of vectors d_i^μ, $i = 0 \ldots p$ and V by c_j^μ, $j = p+1 \ldots d-1$. It turns out useful to distinguish light-like and non-light-like branes (c.f. figure 4.1)

1. If the brane is not tangential to the light-cone, choose the c_i and d_j s.th. :

$$d_i^\mu G_{\mu\nu} d_j^\nu = \eta_{ij}^\parallel \quad c_i^\mu G_{\mu\nu} c_j^\nu = \eta_{ij}^\perp \qquad (4.10)$$

By definition, $d_i^\mu G_{\mu\nu} c_j^\nu = 0$. $\eta^{(\cdot)} = \mathrm{diag}(-1, 1\ldots 1)$ or $= \mathrm{Id}$, if the subspace contains a space-like direction or not.

$$\left(\mathcal{P}_\parallel\right)^\mu{}_\nu \equiv \sum_{i\in D_p} \left(\eta^\parallel\right)^{ii} d_i^\mu d_i^\lambda G_{\lambda\nu} \quad \left(\mathcal{P}_\perp\right)^\mu{}_\nu \equiv \sum_{i\in V_{d-p+1}}^{d} \left(\eta^\perp\right)^{ii} c_i^\mu c_i^\lambda G_{\lambda\nu} \tag{4.11}$$

2. If the brane is tangential to the light-cone let $d_0^\mu \in D_p$ be light-like. Then we choose d_0^ν and c_{p+1}^ν s.th. $d_0^\mu G_{\mu\nu} c_{p+1}^\nu = 1$ which is always possible. All other inner products involving one of the light-like d_0^ν and c_{p+1}^ν with the other basis-vectors should vanish. This means that the other vectors lie in a subspace that is perpendicular to the one spanned by d_0 and c_{p+1} For the remaining vectors we can achieve:

$$d_i^\mu G_{\mu\nu} d_j^\nu = \delta_{ij} \quad i,j = \mathbf{1}\ldots p; \quad c_i^\mu G_{\mu\nu} c_j^\nu = \delta_{ij} \quad i,j = \mathbf{p}+\mathbf{2}\ldots d \tag{4.12}$$

In this case:

$$\begin{aligned} \left(\mathcal{P}_\parallel\right)^\mu{}_\nu &\equiv d_0^\mu c_{p+1}^\lambda G_{\lambda\nu} + \sum_{i=\mathbf{1}}^{p} d_i^\mu d_i^\lambda G_{\lambda\nu} \\ \left(\mathcal{P}_\perp\right)^\mu{}_\nu &\equiv c_{p+1}^\mu d_0^\lambda G_{\lambda\nu} + \sum_{i=\mathbf{p}+\mathbf{2}}^{d} c_i^\mu c_i^\lambda G_{\lambda\nu} \end{aligned} \tag{4.13}$$

We further choose c_{p+1} to be light-like as well (which is achieved by adding λd_0 with appropriate λ). This will ensure the relation $R = R^{-1}$ in (4.15).

The two cases are not continuously connected. *In some of the following formulæ we will omit the indices.* They can be added by taking into account that in matrix-multiplication an upper index is contracted with a lower one. In addition, matrix inversion changes an upper to a lower index and vice versa. Indices are raised and lowered by $G^{\mu\nu}$ and $G_{\mu\nu}$. Then

$$\mathcal{P}_\parallel + \mathcal{P}_\perp = G \tag{4.14}$$

$$\mathcal{P}_\parallel - \mathcal{P}_\perp \equiv R, \quad R \text{ invertible}, \ R = R^{-1} \tag{4.15}$$

The resulting bdy.-conditions are valid even for non-constant G and $\mathcal{F}$ ($\mathcal{F}_\parallel \equiv \mathcal{P}_\parallel^T \mathcal{F} \mathcal{P}_\parallel$):

$$\begin{aligned} \left(\mathcal{P}_\parallel^T G\,\partial_\sigma + \mathcal{F}_\parallel\,\partial_\tau\right) X(\tau,\sigma)\big|_{\sigma\in\partial\mathcal{M}} &= \\ \left(\mathcal{P}_\parallel^T G\,(\partial_+ - \partial_-) + \mathcal{F}_\parallel\,(\partial_+ + \partial_-)\right) X(\tau,\sigma)\big|_{\sigma\in\partial\mathcal{M}} &= 0 \end{aligned} \tag{4.16}$$

$$\mathcal{P}_\perp X(\tau,\sigma)\big|_{\sigma\in\partial\mathcal{M}} = U^i c_i \quad i = p+1\ldots d \quad U^i \in \mathbb{R}, \text{const.} \tag{4.17}$$

$\mathcal{P}_\perp$ is constant for a hyperplane. We can differentiate (4.17) w.r.t. τ. The bdy.-condition can be reformulated as a relation between the left- and right-moving part of the open string. One has to distinguish light-like and other branes. In the non-light-like case $\mathcal{P}^T G = G\mathcal{P}$. Therefore:

$$\begin{aligned} \partial_- X(\tau,\sigma) &= \left(GR - \mathcal{F}_\parallel\right)^{-1}\left(G + \mathcal{F}_\parallel\right)\partial_+ X(\tau,\sigma) \\ &= \left(G - \mathcal{F}_\parallel\right)^{-1}\left(GR + \mathcal{F}_\parallel\right)\partial_+ X(\tau,\sigma) \end{aligned} \quad \text{for } \sigma \in \partial\mathcal{M} \tag{4.18}$$

If $(G \mp \mathcal{F}_{||})$ is not invertible one has a critical case corresponding to vanishing DBI action.[4] This means that $(G \pm \mathcal{F}_{||})$ has a non-trivial kernel. With $v^\mu \mathcal{F}_{\mu\nu} v^\nu = 0$ it is trivial that two such coordinates $v, w \neq 0$ with $(G + \mathcal{F}_{||})v = 0$ and $(G - \mathcal{F}_{||})w = 0$ are light-like and distinct. One can modify the condition (4.18) by inserting suitable projectors, projecting out the light-like v resp. w direction s.th. the matrices $(G \pm \mathcal{F})$ get invertible on the remaining subspaces. In this case the remaining boundary conditions put no obstructions on the left-moving part of the v- and the right-moving part of the w-direction. So the corresponding states will have continuous spectra. This could lead to problems with positivity and unitarity in the quantum theory. The critical case will not be pursued further in this investigation (even though interesting in its own right).

Defining

$$V \equiv \left(G + \mathcal{F}_{||}\right)^{-1}\left(GR - \mathcal{F}_{||}\right) = G^{-1}\left(GR - \mathcal{F}_{||}\right)\left(G + \mathcal{F}_{||}\right)^{-1} G \tag{4.19}$$

we note that V is in $O(1, d-1)$ (i.e. $V^T G V = G$) and (4.18) reduces to:

$$\partial_- X(\tau, \sigma) = V^{-1} \partial_+ X(\tau, \sigma)\,,\; \sigma \in \partial\mathcal{M} \tag{4.20}$$

4.1.1 Open strings with two boundaries

In this subsection we will consider the open string with two boundaries (at $\sigma = 0$ and $\sigma = \pi$), i.e. with two constant but otherwise completely independent U(1) gauge fields F_1 (at $\sigma = 0$) and F_2 (at $\sigma = \pi$). Using (4.4) we will absorb the constant B-field into boundary terms. Defining[5]

$$\mathcal{F}_1 \equiv -B + F_1 \qquad\qquad \mathcal{F}_2 \equiv B + F_2 \tag{4.21}$$

V_1 and V_2 are given by (4.19) with $\mathcal{F}$ substituted by $\mathcal{F}_1$ res. $\mathcal{F}_2$ and R by R_1, R_2. Taking into account the opposite orientation of the $\sigma = 0$ and the $\sigma = \pi$ boundary the two relations between the right and the left moving part of the string are valid:

$$\partial_- X^\nu(\tau, 0) = V_{\mathbf{1}} \partial_+ X^\nu(\tau, 0) \tag{4.22}$$

$$\partial_- X^\mu(\tau, \pi) = V_{\mathbf{2}}^{-1} \partial_+ X^\nu(\tau, \pi). \tag{4.23}$$

Because of the eom (4.8) X^μ can be expanded in the following way:

$$X^\mu = H^\mu + X_L^\mu(\tau + \sigma) + X_R^\mu(\tau - \sigma) \tag{4.24}$$

with H^μ a constant vector. X_L^μ depends only on $\tau + \sigma$ and $\widetilde{X}_R$ only on $\tau - \sigma$. Let us consider the periodicity properties of say $\partial_+ X^\nu$. Note that V is in $O(1, n-1)$ that means it is a Lorentz-transformation. $O(1, n-1)$ consists of four disconnected pieces. For the non-light-like branes one can use $GR - \mathcal{F}_{||} = (G - \mathcal{F}_{||})R$ to see that branes with an even number of transverse dimensions correspond to the $SO(1, n-1)$ subgroup.

[4]For non-light-like branes $\det(G \mp \mathcal{F}_{||}) \neq 0$ is equivalent to $\det(GR \mp \mathcal{F}_{||}) \neq 0$.

[5]The minus sign in front of B takes into account that the direction of the τ derivative at the $\sigma = 0$ end-point is reverse to the derivative at $\sigma = \pi$.

4.1.2 Solution to linear boundary conditions for the cylinder

Defining another w.s.-time $\tilde{\tau} = \tau + \pi$ (π the open string length) the first bdy.-condition (4.22) reads:

$$\underbrace{\partial_- X^\mu(\tilde{\tau} - \pi, 0)}_{=\partial_- X^\mu(\tilde{\tau},\pi)} = \left(V_{\mathbf{1}}\right)^\mu{}_\nu \partial_+ X^\nu(\tau, 0) \tag{4.25}$$

Plugging the left hand side into the second condition (4.23) one gets:

$$\partial_- X^\mu(\tilde{\tau}, \pi) = \left(V_{\mathbf{2}}^{-1}\right)^\mu{}_\nu \underbrace{\partial_+ X^\nu(\tilde{\tau}, \pi)}_{=\partial_+ X^\mu(\tau, 2\pi)} \tag{4.26}$$

Combining the last two equations we see that the following quasi-periodicity holds for $\partial_+ X^\mu$:

$$\partial_+ X^\mu(\tau + \sigma + 2\pi) = \left(V_{\mathbf{2}} V_{\mathbf{1}}\right)^\mu{}_\nu \partial_+ X^\nu(\tau + \sigma) \tag{4.27}$$

As $V_{\mathbf{1},\mathbf{2}}$ (and therefore their product) are in $O(1, n-1) \subset U(1, n-1)$, $V_{\mathbf{2}} V_{\mathbf{1}}$ admits in general $n-1$ complex Eigenvectors $C^\mu_{\lambda_i}$ with Eigenvalues λ ($|\lambda| = 1$) and two real Eigenvalues ξ, ξ^{-1} which belong to two real Eigenvectors C^μ_ξ and $C^\mu_{\xi^{-1}}$.[6] We assume the $C^\mu_{\lambda_i}$ to be "ortho-normalized"(always possible in the case at hand) with respect to the hermitian scalar product $\langle C_1, C_2 \rangle \equiv \sum_\mu C_1^{\mu *} G_{\mu\nu} C_2^\nu$. As $V_{\mathbf{2}} V_{\mathbf{1}}$ is real, for every λ_i, λ_i^* is Eigenvalue (with Eigenvector $C^\mu_{\lambda^*} = C^{\mu *}_\lambda$), too. For $\xi \neq \pm 1$, C^μ_ξ and $C^\mu_{\xi^{-1}}$ are light-like. As their scalar product is non-vanishing, we will normalize C^μ_ξ and $C^\mu_{\xi^{-1}}$ such that $\langle C_{\xi^{-1}}, C_\xi \rangle = 1$. $C_{\xi^{-1}}$, C_ξ are perpendicular to the C_{λ_i}. Let us represent λ_i as

$$\begin{aligned} \lambda_1 = &\quad \exp(i2\pi(\tfrac{-i}{2\pi} \ln \xi) = \exp(i2\pi\theta_1) &(4.28)\\ \lambda_2 = &\quad \exp(i2\pi(\tfrac{i}{2\pi} \ln \xi) = \exp(i2\pi\theta_2) &(4.29)\\ \lambda_i = &\quad \exp(i2\pi\theta_i) &(4.30) \end{aligned}$$

$-1/2 < \theta_i \leq 1/2$, $i = 3 \ldots n$ and $\mathfrak{Im} \ni \theta_{1,2} = \pm \frac{i}{2\pi} \ln \xi$. Denote λ_i^* by λ_{-i} for $i = 3 \ldots n$ and $\lambda_{-1} = \lambda_2$. Similarly $\theta_{-i} = -\theta_i$ represents λ_{-i}. Note however that there are cases without light-like Eigenvectors. For example a pure space rotation could lead (in odd space dimensions) to a time-like Eigenvector with Eigenvalue $\lambda = 1$. The following useful identity holds:

$$\sum_i C^\mu_i C^\nu_{-i} = G^{\mu\nu} \tag{4.31}$$

We will abbreviate the $\lambda_\ell = 1$ Eigenvectors as C_ℓ, $C_\jmath$, etc. With $\partial_\pm X^\mu = \partial_\pm X^\mu_{L/R}$ we see that $X_{L/R}$ is of the form:

$$X^\mu_L = \text{const} + \left(\sum_{\ell : \lambda_\ell = 1} C^\mu_\ell p^\ell (\tau + \sigma) \right) + \frac{\sqrt{\alpha'}}{i\sqrt{2}} \sum_j C^\mu_j \sum_{n_j \in \mathbb{Z} + \theta_j}{}' \frac{a^j_{-n_j}}{n_j} e^{i n_j (\tau + \sigma)} \tag{4.32}$$

---

[6] This is derived in appendix C, where also the other stated facts on the Eigenvectors are proven. We exclude a degenerate case (in which our statement about the Eigenvectors would be wrong and) that might show up for some special light-like Eigenvectors with Eigenvalue $\lambda = \pm 1$, from our further analysis. We shed some light on this case in appendix C as well. For a purely *magnetic* $F_{\mathbf{i}}$-field the transformation $V_{\mathbf{i}}$ is actually a rotation (i.e. $\in O(n)$).

In the following we will absorb the constant part of the X_L and X_R-field into the common constant H^μ (eq. (4.24)). Boundary condition (4.22) prescribes what X_R has to be:

$$\partial_- X_R^\mu(\tau-\sigma,0) = \left(V_{\mathbf{1}}\right)^\mu{}_\nu \partial_+ X_L^\nu(\tau-\sigma,0) \tag{4.33}$$
$$\partial_- X_R^\mu(\tau-\sigma) = \left(V_{\mathbf{1}}\right)^\mu{}_\nu \partial_+ X_L^\nu(\tau+(-\sigma)) \tag{4.34}$$

In the last equation the differentials act on the functions argument. This then leads to:

$$X_R^\mu = \left(\sum_{\ell:\lambda_\ell=1} (V_{\mathbf{1}})^\mu{}_\nu C_\ell^\nu p^\ell(\tau-\sigma)\right) + \frac{\sqrt{\alpha'}}{i\sqrt{2}} \sum_j \left(V_{\mathbf{1}}\right)^\mu{}_\nu C_j^\nu \sum_{n_j\in\mathbb{Z}+\theta_j}{}' \frac{a^j_{-n_j}}{n_j} e^{in_j(\tau-\sigma)} \tag{4.35}$$

Thus we have the following mode expansion of $X^\mu(\tau,\sigma)$:

$$\begin{aligned} X^\mu = \overbrace{H^\mu}^{\text{0-modes}} &+ \overbrace{\sum_{\ell:\lambda_\ell=1} \left((\tau+\sigma)\,\mathrm{Id} + (\tau-\sigma)V_{\mathbf{1}}\right)^\mu{}_\nu C_\ell^\nu p^\ell}^{\text{linear modes}} \\ &+ \underbrace{\frac{\sqrt{\alpha'}}{i\sqrt{2}} \sum_j \sum_{n_j\in\mathbb{Z}+\theta_j}{}' \frac{a^j_{-n_j}}{n_j}\left(e^{in_j(\tau+\sigma)} C_j^\mu + \left(V_{\mathbf{1}}\right)^\mu{}_\nu C_j^\nu e^{in_j(\tau-\sigma)}\right)}_{\text{oscillator-modes}} \end{aligned} \tag{4.36}$$

The Dirichlet condition (4.17) imposes further restriction on the zero-mode part of the string. At $\sigma=0$ the brane (hyperplane) is located at $\mathcal{P}_{\perp,\mathbf{1}}U = U_{\mathbf{1}}$. Then $\mathcal{P}_{\perp,\mathbf{2}}U_{\mathbf{2}} = U_{\mathbf{2}}$ specifies the position of the $\sigma=\pi$ brane. We become aware that

$$\mathcal{P}_{\perp,\mathbf{1}} V_1^{\pm 1} = -\mathcal{P}_{\perp,\mathbf{1}} \qquad\qquad \mathcal{P}_{\perp,\mathbf{2}} V_2^{\pm 1} = -\mathcal{P}_{\perp,\mathbf{2}} \tag{4.37}$$

as well as

$$V_1 C_{\lambda_\ell} = V_2^{-1} C_{\lambda_\ell} \quad \text{ for } \lambda_\ell = 1 \tag{4.38}$$

with λ_ℓ the Eigenvalue of V_2V_1. Let us rewrite the zero- and linear-modes of X:

$$X_0^\mu = H^\mu + \left(\sum_{\ell:\lambda_\ell=1} \left(\tau\left(\mathrm{Id}+V_{\mathbf{1}}\right)^\mu{}_\nu + \sigma\left(\mathrm{Id}-V_{\mathbf{1}}\right)^\mu{}_\nu\right) C_\ell^\nu p^\ell\right) \tag{4.39}$$

4.1.2.1 World sheet momentum and Hamiltonian

In order to quantize the string one has to know the (gauge-dependent) canonical momentum:

$$\begin{aligned} P^\mu(\sigma) &= \frac{\partial}{\partial \dot{X}_\mu} L(X,\partial X) \\ &= \frac{1}{2\pi\alpha'}\left(\dot{X}^\mu + B^\mu{}_\nu X'^\nu - \left(\delta(\sigma)A_{\mathbf{1}}^\mu + \delta(\sigma-\pi)A_{\mathbf{2}}^\mu\right)\right) &(4.40) \\ &= \frac{1}{2\pi\alpha'}\left(\dot{X}^\mu + B^\mu{}_\nu X'^\nu + \frac{1}{2}\left(\delta(\sigma)F_{\mathbf{1}}{}^\mu{}_\nu X^\nu + \delta(\sigma-\pi)F_{\mathbf{2}}{}^\mu{}_\nu X^\nu\right)\right) &(4.41) \end{aligned}$$

Equation (4.40) is valid for all $U(1)$-field strengths, (4.41) only for constant field strengths in our particular gauge. The momentum is not conserved since the Lagrangian varies under pure translations.[7]

Another important quantity is the world sheet Hamiltonian:

$$
\begin{aligned}
H &= \int_0^\pi d\sigma\, P^\mu(\sigma)\dot{X}_\mu - L(\sigma) = \frac{1}{4\pi\alpha'}\int_0^\pi d\sigma\, \dot{X}^\mu\dot{X}_\mu + X'^\mu X'_\mu \\
&= \frac{1}{2\pi\alpha'}\int_0^\pi d\sigma\, \partial_+X^\mu\partial_+X_\mu + \partial_-X^\mu\partial_-X_\mu = \frac{1}{\pi\alpha'}\int_0^\pi d\sigma\, \partial_+X^\mu\partial_+X_\mu
\end{aligned} \tag{4.42}
$$

H is gauge invariant in terms of the X^μ's and its derivatives. With

$$
\partial_+X_0^\mu = C_\ell^\mu p^\ell \tag{4.43}
$$

the momentum-mode part of the Hamiltonian is:

$$
H_{\text{lin}} = \frac{1}{\alpha'}\underbrace{C_\jmath^\mu G_{\mu\nu}C_\ell^\nu}_{\equiv G_{\jmath\ell}}\, p^\jmath p^\ell \tag{4.44}
$$

Similarly one obtains the oscillator-part (which still has to be normal ordered):

$$
H_{\text{osc.}} = \frac{1}{2}\sum_k \sum_{n_k\in\mathbb{Z}+\theta_k}{}' \underbrace{C_{-k}^\mu G_{\mu\nu}C_k^\nu}_{\equiv G_{-k\,k}}\, a_{n_k}^{-k}a_{-n_k}^k \tag{4.45}
$$

4.2 Quantization of open strings with linear boundary conditions

In the following we will quantize the classical solution. To do this, we first calculate the *canonical two-form* $\Omega(P,X)$ in terms of the classical solution which we restrict in a second step to a subspace on which $\Omega(P,X)$ is nondegenerate. The inverse of $\Omega(P,X)$ (on this subspace) defines the *Poisson-bracket*. By substitution of the Poisson bracket by $-i$ times the commutator we perform the transition to the quantized string. Alternatively one could have tried to implement the boundary conditions via a Dirac-bracket. However we think that the method applied here is more direct and less complicated.

[7] By momentum we mean also the integral $\int d\sigma P$.

4.2.1 Canonical two-form and canonical quantization

In order to quantize the system lets look at the canonical two-form[8]

$$\Omega(P,X) = \int_0^\pi d\sigma\ DP^\mu \wedge DX_\mu \tag{4.46}$$
$$= \frac{1}{2\pi\alpha'}\int_0^\pi d\sigma\ \Big(D\dot{X}^\mu \wedge DX_\mu$$
$$+ \tfrac{1}{2}(\delta(\sigma)F_1 + \delta(\sigma-\pi)F_2)^\mu{}_\nu DX^\nu \wedge DX_\mu + \tfrac{1}{2}B^\mu{}_\nu \frac{\partial}{\partial\sigma}\big(DX^\nu \wedge DX_\mu\big)\Big)$$
$$= \frac{1}{2\pi\alpha'}\int_0^\pi d\sigma\ \Big(D\dot{X}^\mu \wedge DX_\mu$$
$$+ \tfrac{1}{2}\left(\delta(\sigma)\mathcal{F}_{\mathbf{1}} + \delta(\sigma-\pi)\mathcal{F}_{\mathbf{2}}\right)^\mu{}_\nu\, DX^\nu \wedge DX_\mu\Big) \tag{4.47}$$

which is time independent in case of constant field strengths, two-form potential B and on-shell X^μ:[9]

$$\frac{d}{d\tau}\Omega(P,X) =$$
$$\frac{1}{2\pi\alpha'}\int_0^\pi d\sigma\big(\overbrace{D\ddot{X}^\mu}^{=DX''^\mu} \wedge DX_\mu + (\delta(\sigma)\mathcal{F}_1 + \delta(\sigma-\pi)\mathcal{F}_2)^\mu{}_\nu D\dot{X}^\nu \wedge DX_\mu\big)$$
$$= \frac{1}{2\pi\alpha'}\int_0^\pi d\sigma\, d\left(DX'^\mu \wedge DX_\mu\right) + \underbrace{DX'^\mu \wedge DX'_\mu}_{=0}$$
$$+ \big(\delta(\sigma)\mathcal{F}_{\parallel\,\mathbf{1}} + \delta(\sigma-\pi)\mathcal{F}_{\parallel\,\mathbf{2}}\big)^\mu{}_\nu D\dot{X}^\nu \wedge DX_\mu = 0$$

Therefore one can neglect the τ-dependent parts of Ω.

4.2.1.1 Quantization of zero- and linear-modes

The following two expressions are useful to follow the discussion:

$$\dot{X}_0^\mu = (G+V_1)^\mu{}_\nu C_\ell^\nu p^\ell \qquad X^\mu_{\tau\text{-indep.}} = H^\mu + \sigma(G-V_1)^\mu{}_\nu C_\ell^\nu p^\ell \tag{4.48}$$

The Poisson bracket is the inverse of the restriction $\Omega|_U$ of the canonical two-form Ω to a subspace U s.th. $\Omega|_U$ is invertible. To shorten the notation, we will not write down the derivative D explicitly. We will now quantize the individual

[8] $D\,P$ and $D\,X$ means the derivative with respect to the target space. Therefore Ω is manifestly $U(1)$-gauge invariant since $P \to P + D\,\xi$ and X is invariant.

[9] In the last line only parallel components of the $\mathcal{F}$ fields contribute, as the perpendicular components of $\dot{X}^\mu$ vanish at boundary.

(zero) modes:[10]

$$
\begin{aligned}
&\Omega_0(P,X)\\
&= \int_0^\pi d\sigma \left(\overbrace{\frac{1}{2\pi\alpha'}\Big((G+V_1)_{\mu\nu}C^\nu_\ell p^\ell \wedge H^\mu + \sigma C^\mu_\ell (V_1^T - V_1)_{\mu\nu} C^\nu_\jmath p^\ell \wedge p^\jmath\Big)}^{\text{from } \dot{X}^\mu \wedge X_\mu} \right)\\
&\qquad + \frac{1}{4\pi\alpha'}\big(\mathcal{F}_{\parallel\mathbf{1}} + \mathcal{F}_{\parallel\mathbf{2}}\big)_{\mu\nu} H^\nu \wedge H^\mu\\
&+ \frac{1}{2\alpha'}\big(\mathcal{F}_{\mathbf{2}}(G-V_1)\big)_{\mu\nu} C^\nu_\ell p^\ell \wedge H^\mu + \frac{\pi}{4\alpha'} C^\mu_\jmath \big((G-V_1^T)\mathcal{F}_{\mathbf{2}}(G-V_1)\big)_{\mu\nu} C^\nu_\ell\, p^\ell \wedge p^\jmath
\end{aligned}
$$

Summarizing,

$$
\begin{aligned}
\Omega_0(P,X) = {}& \frac{1}{4\pi\alpha'}\big(\mathcal{F}_{\parallel\mathbf{1}} + \mathcal{F}_{\parallel\mathbf{2}}\big)_{\mu\nu} H^\nu \wedge H^\mu\\
& + \frac{1}{2\alpha'}\Big((G+V_1) + \big(\mathcal{F}_{\mathbf{2}}(G-V_1)\big)\Big)_{\mu\nu} C^\nu_\ell p^\ell \wedge H^\mu\\
& + \frac{\pi}{4\alpha'} C^\mu_\jmath \big((G-V_1^T)\mathcal{F}_{\mathbf{2}}(G-V_1) + (V_1 - V_1^T)\big)_{\mu\nu} C^\nu_\ell\, p^\ell \wedge p^\jmath
\end{aligned}
\tag{4.49}
$$

Now we will restrict to an invertible subspace. We introduce a system of vectors e_i^μ such that

$$
(A_1)_{ij} \equiv \frac{1}{4\pi} e_i^\mu \big(\mathcal{F}_{\parallel\mathbf{1}} + \mathcal{F}_{\parallel\mathbf{2}}\big)_{\mu\nu} e_j^\nu \qquad i,\, j = 1 \dots p \tag{4.50}
$$

is invertible (such a system, (including possibly the empty set) always exists as $\mathcal{F}_{\parallel\mathbf{i}}$ is anti-symmetric). Then we define H^i and H^a by:

$$
H^\mu = e_i^\mu H^i + d_a^\mu H^a \tag{4.51}
$$

The d_a^μ are made orthogonal to the e_i^μ. H^μ is real as the string coordinate X^μ is real. This leads to some restrictions on the phase of the H^i and H^a. In the quantized version of the string X^μ becomes a hermitian operator. The classical restrictions on the H^i and H^a lead to some restrictions on their properties as operators.

In some special cases (i.e. when Dirichlet conditions are absent), the e_i^μ are chosen to be $(G+\mathcal{F}_1)C_i$ with $\lambda_i \neq 1$. This leads to the simple relation $H^{-i} = H^{i\,\dagger}$. The analogous definition $d_a \equiv (G+\mathcal{F}_1)C_a$ with $\lambda_a = 1$ forces the corresponding H^a to be hermitian in this case.

With this choice and taking into account (4.49) the general form of $\Omega_0|_u$ is (in matrix form):

$$
\Omega_0|_U = \frac{1}{\alpha'} \begin{pmatrix} A_1 & 0 & K \\ 0 & 0 & N \\ -K^T & -N^T & A_2 \end{pmatrix} \tag{4.52}
$$

[10] The perpendicular components of $B_{\mu\nu}$ in the term $\propto \big(\mathcal{F}_{\mathbf{1}} + \mathcal{F}_{\mathbf{2}}\big)_{\mu\nu} H^\nu \wedge H^\mu$ cancel.

By comparison with (4.49) we identify in addition to (4.50):

$$
\begin{aligned}
(K)_{i\ell} &\equiv \frac{1}{4} e_i^\mu \Big((G+V_1) + \big(\mathcal{F}_{\mathbf{2}}(G-V_1)\big) \Big)_{\mu\nu} C_\ell^\nu \\
(N)_{a\ell} &\equiv \frac{1}{4} d_a^\mu \Big((G+V_1) + \big(\mathcal{F}_{\mathbf{2}}(G-V_1)\big) \Big)_{\mu\nu} C_\ell^\nu \qquad (4.53)\\
(A_2)_{\ell\jmath} &\equiv \frac{\pi}{4} C_\jmath^\mu \big((G-V_1^T)\mathcal{F}_{\mathbf{2}}(G-V_1) + (V_1 - V_1^T) \big)_{\mu\nu} C_\ell^\nu
\end{aligned}
$$

$$
\text{with } i = 1\ldots p; \qquad a,\, \ell,\, \jmath = p+1\ldots r \qquad (4.54)
$$

While p is the dimension of $\operatorname{im}\big(\mathcal{F}_{\|\mathbf{1}} + \mathcal{F}_{\|\mathbf{2}}\big)$ (or equivalently: the dimension of A_1), $(r-p)$ is the dimension of its kernel. In (4.54) we already used that the matrix N has to be a square matrix. This is true for the following reasons:

1. In order for $\Omega_0|_U$ to be invertible, N^T has to be of maximal rank (i.e. $\dim A_2$). Therefore the column number of N equals the number of p^ℓ dofs (or equivalently: the number of Eigenvalue $\lambda = 1$ Eigenvectors of V_2V_1).
2. In order for (4.52) to be invertible, its determinant has to be $\neq 0$. If the number of rows in N exceeds the dimension of the square matrix A_2, at least two column vectors of $\Omega_0|_U$ would be linear dependent. As a consequence the determinant would vanish which contradicts the invertibility of $\Omega_0|_U$. As $\Omega_0|_U$ is invertible by definition, N must be a square matrix.
3. As N has maximal rank, it is invertible.

In other words: the space spanned by the d_a^μ has to be reduced such that N gets invertible which is needed to ensure invertibility of $\Omega_0|_U$. While A_1 and A_2 are antisymmetric, K is in general not a square matrix.[11] In this notation

$$
\Omega_0(P,X)|_U = x_i(\Omega_0)_{ij} \wedge x_j, \qquad \vec{x} = (H^i, H^a, p^\jmath) \qquad (4.55)
$$

To get the Poisson bracket of the bosonic zero-modes one has to invert Ω_0:

$$
\{x^i, x^j\}_{\text{P.B.}} = \frac{1}{2}(\Omega_0^{-1})^{ij} \qquad (4.56)
$$

The solution is

$$
\Omega_0^{-1} = \alpha' \begin{pmatrix} A_1^{-1} & -A_1^{-1}KN^{-1} & 0 \\ -\big(N^T\big)^{-1} K^T A_1^{-1} & \big(N^T\big)^{-1}(A_2 - K^T A_1^{-1} K)N^{-1} & -\big(N^T\big)^{-1} \\ 0 & N^{-1} & 0 \end{pmatrix} \qquad (4.57)
$$

As Ω_0 is a symplectic form it can be transformed into a more convenient form by a general linear transformation of the zero modes H^i, H^a, $p^\jmath$.

$$
\tilde{\Omega}_0 = S^T \Omega_0 S = \frac{1}{\alpha'} \begin{pmatrix} A_1 & 0 & 0 \\ 0 & 0 & N \\ 0 & -N^T & 0 \end{pmatrix} \qquad (4.58)
$$

[11] We found that the dimensions of the spaces $\langle d_a \rangle$ and $\langle C^\jmath; \lambda_\jmath = 1 \rangle$ is equal. For the specific situation where the Dirichlet directions are the same for both branes we can even identify: $d^\jmath = C^\jmath$.

with

$$S = \begin{pmatrix} 1 & 0 & 0 \\ -\left(N^T\right)^{-1} K^T & 1 & \left(N^T\right)^{-1} A_2/2 \\ 0 & 0 & 1 \end{pmatrix} \tag{4.59}$$

The transformed zero modes $\tilde{H}^i$, $\tilde{H}^\ell$, $\tilde{p}^\jmath$ are defined by:

$$\begin{pmatrix} \tilde{H}^i \\ \tilde{H}^a \\ \tilde{p}^\jmath \end{pmatrix} = S^{-1} \begin{pmatrix} H^i \\ H^a \\ p^\jmath \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ \left(N^T\right)^{-1} K^T & 1 & -\left(N^T\right)^{-1} A_2/2 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} H^i \\ H^a \\ p^\jmath \end{pmatrix} \tag{4.60}$$

The $\tilde{H}^i$, $\tilde{H}^a$, $\tilde{p}^\jmath$ have rather simple Poisson-brackets and commutators (all others vanishing):[12]

$$\begin{aligned} \left\{\tilde{H}^i, \tilde{H}^k\right\} =& \frac{\alpha'}{2}\left(A_1^{-1}\right)^{ik} \\ \left\{\tilde{H}^a, \tilde{p}^\jmath\right\} =& -\frac{\alpha'}{2} N^{a\jmath} \end{aligned} \quad \xrightarrow{\{.,.\}\to -i[.,.]} \quad \begin{aligned} \left[\tilde{H}^i, \tilde{H}^k\right] =& \frac{\alpha'}{2i}\left(A_1^{-1}\right)^{ik} \\ \left[\tilde{H}^a, \tilde{p}^\jmath\right] =& i\frac{\alpha'}{2}\left(N^{-1}\right)^{a\jmath} \end{aligned} \tag{4.61}$$

4.2.1.2 Quantization of oscillator modes

Now we look at the oscillator part of Ω, which is also time independent. For simplicity we split $\Omega_{\text{osc.}}$ into $\Omega_1 + \Omega_2$ and Ω_3 where $\Omega_1 + \Omega_2 \propto \dot{X} \wedge X$ is from the bulk integral and Ω_3 is the boundary term.

$$\begin{aligned} &\frac{1}{2\pi\alpha'} \dot{X}^\mu_{\text{osc.}}(\tau,\sigma) \wedge X_{\text{osc.}\,\mu}(\tau,\sigma) \\ &\quad = \frac{1}{2\pi\alpha'} \frac{-i\alpha'}{2} \cdot \sum_{j,\,l} \sideset{}{'}\sum_{\substack{n_j\in\mathbb{Z}+\theta_j \\ m_l\in\mathbb{Z}+\theta_l}} e^{i\tau(n_j+m_l)} \left(G e^{in_j\sigma} + V_1 e^{-in_j\sigma}\right)^\mu{}_\epsilon C^\epsilon_j \\ &\qquad\qquad G_{\mu\nu}\left(G e^{im_l\sigma} + V_1 e^{-im_l\sigma}\right)^\nu{}_\kappa C^\kappa_l \frac{a^j_{-n_j} \wedge a^l_{-m_l}}{m_l} \end{aligned} \tag{4.62}$$

As we consider only the τ-independent terms, we impose $n_j = -m_l$, $j = -l$.

$$\begin{aligned} &\frac{1}{2\pi\alpha'} \dot{X}^\mu_{\text{osc.}}(\tau,\sigma) \wedge X_{\text{osc.}\,\mu}(\tau,\sigma) = \frac{-i}{4\pi} \sum_l \sideset{}{'}\sum_{m_l\in\mathbb{Z}+\theta_l} \\ &\left(C^\epsilon_{-l} \underbrace{\left(G e^{-im_l\sigma} + V_1^T e^{im_l\sigma}\right)_\epsilon{}^\mu \left(G e^{im_l\sigma} + V_1 e^{-im_l\sigma}\right)_{\mu\kappa}}_{=(2G + {V_1}^{-1}\cdot\exp(i2m_l\sigma) + V_1\cdot\exp(-i2m_l\sigma))_{\epsilon\kappa}} C^\kappa_l\right) \frac{a^{-l}_{m_l} \wedge a^l_{-m_l}}{m_l} \end{aligned} \tag{4.63}$$

[12] $N^{a\jmath}$ is the inverse transposed of $N_{\imath b}$

From this we get $\Omega_1 + \Omega_2$:

$$\Omega_1 + \Omega_2 = \frac{1}{2\pi\alpha'} \int_0^\pi d\sigma \dot{X}^\mu_{\text{osc.}} \wedge X_{\text{osc.}\,\mu}$$

$$= \overbrace{\frac{-i}{2} \sum_l G_{-l,l} \sum_{m_l \in \mathbb{Z}+\theta_l}{}^{\prime} \frac{a^{-l}_{m_l} \wedge a^l_{-m_l}}{m_l}}^{\equiv \Omega_1}$$

$$\underbrace{- \sum_l \sum_{m_l \in \mathbb{Z}+\theta_l}{}^{\prime} \frac{1}{8\pi} C^\mu_{-l} \Big(\overbrace{V_1^{-1}(\lambda_l - G)}^{\lambda_l C^T_{-l} = ((V_2V_1)^{-1}C_{-l})^T} - \overbrace{V_1(\lambda_{-l} - G)}^{V_1\lambda_{-l}C_l = V_2^{-1}C_l} \Big)_{\mu\nu} C^\nu_l \frac{a^{-l}_{m_l} \wedge a^l_{-m_l}}{m_l^2}}_{\equiv \Omega_2} \tag{4.64}$$

$$\Omega_2 = -\frac{1}{8\pi} \sum_{\substack{l \\ m_l \in \mathbb{Z}+\theta_l}}{}^{\prime} C^\mu_{-l} \Big(\big(V_2 - V_2^{-1}\big) + \big(V_1 - V_1^{-1}\big) \Big)_{\mu\nu} C^\nu_l \frac{a^{-l}_{m_l} \wedge a^l_{-m_l}}{m_l^2} \tag{4.65}$$

Next we will show that $\Omega_2 + \Omega_3$ vanishes. If we assume that this is true we immediately obtain the Poisson-bracket for the modes (all others vanishing):

$$\big\{ a^l_{-m_l}, a^{-l}_{m_l} \big\} = i m_l G^{-l\,l} \tag{4.66}$$

Ω_3 is the $d\sigma$-integral over:

$$\frac{1}{4\pi\alpha'} \left(\delta(\sigma)(\mathcal{F}_{\mathbf{1}})_{\mu\nu} X^\nu_{\text{osc.}} + \delta(\sigma - \pi)(\mathcal{F}_{\mathbf{2}})_{\mu\nu} X^\nu_{\text{osc.}} \right) \wedge X_{\text{osc.}\,\mu}$$

$$= -\frac{1}{8\pi} \left(\delta(\sigma)(\mathcal{F}_{\mathbf{1}})_{\mu\nu} + \delta(\sigma - \pi)(\mathcal{F}_{\mathbf{2}})_{\mu\nu} \right) e^{i\tau(n_j + m_l)}$$

$$\sum_{j,\,l} \sum_{\substack{n_j \in \mathbb{Z}+\theta_j \\ m_l \in \mathbb{Z}+\theta_l}}{}^{\prime} \big(G e^{i n_j \sigma} + V_1 e^{-i n_j \sigma} \big)^\nu{}_\epsilon C^\epsilon_j \big(G e^{i m_l \sigma} + V_1 e^{-i m_l \sigma} \big)^\mu{}_\kappa C^\kappa_l \frac{a^j_{-n_j} \wedge a^l_{-m_l}}{n_j m_l} \tag{4.67}$$

We only need to consider the τ-independent terms ($n_j = -m_l$). Using again $\lambda_l C^T_{-l} = C^T_{-l}(V_2V_1) = C^T_{-l} V_2 (V_1^T)^{-1}$ we can split Ω_3 into $\mathcal{F}_{\|\,\mathbf{1}}$ respectively $\mathcal{F}_{\|\,\mathbf{2}}$ dependent terms:

$$\Omega_3 = -\frac{1}{8\pi} \sum_l \Sigma_{-l,l} \sum_{m_l \in \mathbb{Z}+\theta_l}{}^{\prime} \frac{a^{-l}_{m_l} \wedge a^l_{-m_l}}{m_l^2} \tag{4.68}$$

where we have defined:[13]

$$\begin{aligned} \Sigma_{-l,l} &= C^\mu_{-l} \Big(\big(G + V_1^T\big) \mathcal{F}_{\|\,\mathbf{1}} (G + V_1) + \big(G + \lambda_l V_1^T\big) \mathcal{F}_{\|\,\mathbf{2}} \big(G + V_1 \lambda_l^{-1}\big) \Big)_{\mu\nu} C^\nu_l \\ &= C^\mu_{-l} \Big(\big(G + V_1^{-1}\big) \mathcal{F}_{\|\,\mathbf{1}} (G + V_1) + (G + V_2) \mathcal{F}_{\|\,\mathbf{2}} \big(G + V_2^{-1}\big) \Big)_{\mu\nu} C^\nu_l \end{aligned} \tag{4.69}$$

[13] We have projected onto parallel components of the $\mathcal{F}_{\mathbf{i}}$-fields, because $G + V_i$ contains the projector $\mathcal{P}_{\|,i}$, that removes the perpendicular components.

As Ω_2 and Ω_3 are symmetric w.r.t. an exchange $\mathcal{F}_{\|\mathbf{1}} \leftrightarrow \mathcal{F}_{\|\mathbf{2}}$ we consider only one part, e.g. the $\mathcal{F}_{\|\mathbf{1}}$ dependent part:

$$\begin{aligned}
&(\Omega_2+\Omega_3)|_{\mathcal{F}_{\|\mathbf{1}}} \\
&= -\frac{1}{8\pi}\sum_l C^\mu_{-l}\big(G+V_1^T\big)\overbrace{\big(\mathcal{F}_{\|\mathbf{1}}(G+V_1)-(G-V_1)\big)_{\mu\nu}}^{(*)} C^\nu_l \sum_{m_l\in\mathbb{Z}+\theta_l}{}' \frac{a^{-l}_{m_l}\wedge a^l_{-m_l}}{m_l^2}
\end{aligned} \tag{4.70}$$

$(*)$ vanishes: The matrix R_1 that determines the Dirichlet conditions on the first brane, commutes with V_1 and $\mathcal{F}_{\|\mathbf{1}}$:

$$(*) = \big(\mathcal{F}_{\|\mathbf{1}}(G+V_1)-(G-V_1)\big)_{\mu\nu} = R\big(2\mathcal{F}_{\|\mathbf{1}}-2\mathcal{F}_{\|\mathbf{1}}\big)\big(G+\mathcal{F}_{\|\mathbf{1}}\big)^{-1} = 0 \tag{4.71}$$

The $\mathcal{F}_{\|\mathbf{2}}$ dependent terms cancel analogously. Therefore $\Omega_2+\Omega_3$ vanishes and we end up with the Poisson brackets (4.56), (4.57) and (4.66) for the zero-, linear- and oscillator-modes. The commutators that are obtained by the substitution $\{\,.\,,\,.\,\}\rightarrow\frac{1}{i}[\,.\,,\,.\,]$ are (all others vanishing):

$$\big[a^l_{-m_l}, a^{-l}_{m_l}\big] = m_l G^{-l\,l} \tag{4.72}$$

$$\big[H^\jmath, p^\ell\big] = i\frac{\alpha'}{2}\big(N^{-1}\big)^{\jmath\ell} \tag{4.73}$$

$$\big[H^\jmath, H^\ell\big] = i\frac{\alpha'}{2}\Big(\big(N^T\big)^{-1}\big(K^T A_1^{-1}K - A_2\big)N^{-1}\Big)^{\jmath\ell} \tag{4.74}$$

$$\big[H^j, H^k\big] = \frac{\alpha'}{2i}\big(A_1^{-1}\big)^{jk} \tag{4.75}$$

$$\big[H^j, H^\ell\big] = i\frac{\alpha'}{2}\big(A_1^{-1}KN^{-1}\big)^{j\ell} \tag{4.76}$$

We observe that in contrast to the zero and linear modes the quantization of the oscillator modes is not affected by the B-field.

As an application we will calculate the commutator $[X(\tau,\sigma),X(\tau,\sigma')]$ at the string end-points for the case without Dirichlet conditions. It turns out that this commutator is ill-defined for $\sigma=\sigma'=0,\pi$ and that it has to be regularized.

4.2.1.3 Quantization of zero and momentum modes in toroidal compactifications

We already noted that the canonical momentum (4.41) is *not* a constant of motion, even though the field strengths F do not depend on space. This is due to the fact that the Lagrangian (4.2) contains the vector potential A. Therefore $S_{\text{bos, bdy.}}$ is explicitly space dependent for any nontrivial NS field strength F:

$$X_\mu \rightarrow X_\mu + \delta X_\mu \qquad \Rightarrow \qquad A_\mu \rightarrow A_\mu + \delta X^\nu \partial_\nu A_\mu \tag{4.77}$$

In the gauge chosen ($A_\nu = \frac{X^\mu}{2}F_{\mu\nu}$) we note however that the combination of a translation δX and a gauge transformation $A\rightarrow A-\partial\chi$ with $\chi = \frac{\delta X^\mu}{2}F_{\mu\nu}X^\nu$

leaves the action invariant. We consider the following "generalized momentum":

$$\Pi^\mu = P^\mu + \frac{1}{4\pi\alpha'}\big(\delta(\sigma)F_{\mathbf{1}}{}^{\mu}{}_{\nu}X^\nu + \delta(\sigma-\pi)F_{\mathbf{2}}{}^{\mu}{}_{\nu}X^\nu\big) \tag{4.78}$$

$$= \frac{1}{2\pi\alpha'}\Big(\dot{X}^\mu + B^\mu{}_\nu X'^\nu + \big(\delta(\sigma)F_{\mathbf{1}}{}^{\mu}{}_{\nu}X^\nu + \delta(\sigma-\pi)F_{\mathbf{2}}{}^{\mu}{}_{\nu}X^\nu\big)\Big) \tag{4.79}$$

In contrast to the canonical momentum (4.41), $\int d\sigma \Pi^\mu(\tau,\sigma)$ is a constant of motion. It generates combined translations and gauge transformations. For $F = 0$ it reduces to the ordinary canonical momentum, which is conserved in the $F = 0$ case. Therefore we interpret (4.78) as a generalization of the generators for translations. This is very similar to the magnetic translation group introduced in condensed matter physics (cf. [93, 94]). In string theory the momentum (4.78) already showed up in [95]. For simplicity, we consider the case without Dirichlet conditions. Inserting the solution (4.36) into (4.79) and integrating $\Pi^\mu(\tau,\sigma)$ over σ we end up with the τ independent expression:

$$\Pi = \frac{F_1+F_2}{2\pi\alpha'}H + \frac{1}{\alpha'}\big(G+\mathcal{F}_2\mathcal{F}_1\big)\big(G+\mathcal{F}_1\big)^{-1}\sum_{\ell\,:\,\lambda_\ell=1} C_\ell p^\ell \tag{4.80}$$

We use some of the results of the following section 4.3.1 to further simplify this expression. Especially the relations (4.117) and (4.123) turn out to be useful. Using in addition $\big(G+\mathcal{F}_1\big)^{-1}C_\ell = \big(G-\mathcal{F}_2\big)^{-1}C_\ell$ (for $\lambda_\ell = 1$), which can be deduced from the characteristic polynomial (4.106), we can rewrite the *magnetic translation generator* Π:[14]

$$\Pi = \frac{F_1+F_2}{2\pi\alpha'}\sum_{k\,:\,\lambda_k\neq 1} C_k H^k + \Big(G \mp \mathcal{F}_{\substack{1\\2}}\Big)\sum_{\ell\,:\,\lambda_\ell=1} C_\ell p^\ell \tag{4.81}$$

We calculate the following commutator of the magnetic translation operators Π^μ:

$$\big[\Pi^\mu, \Pi^\nu\big] = \frac{i}{2\pi\alpha'}(F_1+F_2)^{\mu\nu} \tag{4.82}$$

Finite translations T by a vector R_μ are generated via the exponential map:

$$T(R) \equiv e^{i\langle \Pi, R\rangle} \tag{4.83}$$

By using the Campbell-Hausdorff Formula we get the following multiplication law for the algebra of magnetic translations:

$$T\big(R^{(1)}\big)T\big(R^{(2)}\big) = T\big(R^{(2)}\big)T\big(R^{(1)}\big)e^{\frac{i}{2\pi\alpha'}R^{(1)}(F_1+F_2)R^{(2)}} \tag{4.84}$$

Furthermore, the magnetic translation group is associative. (This justifies the name "group". The inverse of $T(R)$ is $T(-R)$.)

We will now have a closer look to the restrictions that arise if we compactify the theory on a d-torus. Strings on a torus generated by a lattice $2\pi\Gamma^d$ (cf. section 2.2, eq. (2.17)) are described by wave functions that are invariant under

[14]This term is used as Π is very similar to the generators of the group introduced in [93,94].

all translations $R^{(i)} \in 2\pi\Gamma^d$. This implies that the commutator of two such translations $T\big(R^{(1)}\big)$ and $T\big(R^{(2)}\big)$ has to vanish:

$$T\big(R^{(1)}\big)T\big(R^{(2)}\big) - T\big(R^{(1)}\big)T\big(R^{(2)}\big) = 0 \quad \forall\, e^{(1)}, e^{(2)} \in \Gamma^d$$
$$\Longrightarrow e^{(1)} \frac{(F_1+F_2)}{\alpha'} e^{(2)} \in \mathbb{Z} \quad (4.85)$$

This condition is equivalent to the condition that[15]

$$\int_{\mathcal{C}} c_1\big((F_1+F_2)/\alpha'\big) = \int_{\mathcal{C}} \frac{F_1+F_2}{\alpha'} \in 4\pi\mathbb{Z} \quad \forall \mathcal{C} \in C_2 \tag{4.86}$$

The normalization in the above formula is unconventional, but one observes that the result is also a consequence of the mathematical observation that the first Chern class of a $U(1)$-connection on a d-torus takes values in $\mathbb{Z}^{\binom{d}{2}}$.[16] The matrix $1/\alpha'(F_1+F_2)$ is an antisymmetric map over the free module $\mathbb{Z}^d$. According to [96], chapter XIV, the corresponding matrix can be brought to a block diagonal form by a change of base:

$$\frac{1}{\alpha'} S^T(F_1+F_2)S = \bigoplus_{j=1}^{\lceil d/2 \rceil} \begin{pmatrix} 0 & f^{(j)} \\ -f^{(j)} & 0 \end{pmatrix} \,, \quad f^{(j)} \in \mathbb{Z},\ S \in SL(d,\mathbb{Z}) \tag{4.87}$$

The above sum does *not* need to be an orthogonal one, because in general there does not exist a change in basis described by S that is contained in $SO(d,\mathbb{Z})$ leading at the same time to the block-diagonal form (4.87).

In section 4.3.1 we will see, that the space spanned by C_l with $\lambda_l \neq 1$ is perpendicular to the one spanned by C_ℓ with $\lambda_\ell = 1$. Furthermore the projection on these spaces splits the field strengths F_i according to eq. (4.137) and eq. (4.138) (p. 112). We write the torus lattice Γ^d as a direct sum (which is not necessarily an orthogonal sum[17]):

$$\Gamma^d = \Gamma^{d-p} \oplus \Gamma^p \tag{4.88}$$

$d-p$ is the number of Eigenvectors with Eigenvalue $\lambda_\ell = 1$, that lie in the lattice Γ^d. p is the number of Eigenvalues $\lambda_i \neq 1$, with Eigenvectors lying in the (complexified) space that is isomorphic (though not necessarily identical) to the $\mathbb{C}$-vector space spanned Γ^p. Note however that the orthogonal split described by eq. (4.137) and eq. (4.138) is in general not compatible with the decomposition (4.87). Only the space spanned over $\mathbb{R}$ by the Eigenvectors C_ℓ is identical with the space spanned by the kernel of (4.87). (The kernel of a matrix is unique, and C_ℓ spans the kernel of (F_1+F_2).)

---

[15] C_2 denotes the group of two-cycles on the torus $T^d = \mathbb{R}^d/\Gamma^d$. $c_1(F_1+F_2)$ is the first Chern class w.r.t. the $U(1)$ field strength F_1+F_2.

[16] $\binom{d}{2}$ is the dimension of $H_2(T^d)$.

[17] This means that two lattice vectors $v_1 \in \Gamma^p$ and $v_2 \in \Gamma^{d-p}$ might have non-vanishing scalar product.

Requiring that $T\big(2\pi\Gamma^{d-p}\big)$ is represented trivially on all Eigenfunctions of the Hamiltonian[18] leads via (4.83) to the condition:

$$\Pi \in \Gamma^{*\,(d-p)} \tag{4.89}$$

It is very convenient, though not necessary, to choose the C_ℓ to form a basis of lattice $\Gamma^{(d-p)}$. Writing the vector Π in the dual basis: $\Pi = m_\jmath e^\jmath$ with $e^\jmath \in \Gamma^{*\,(d-p)}$ we obtain:[19]

$$\sum_\jmath e_\jmath p^\jmath(\vec{m}) = \Big(G \mp \mathcal{F}_{\frac{1}{2}}\Big)^{-1} \sum_\jmath m_\jmath e^\jmath \qquad \vec{m} \in \mathbb{Z}^{(d-p)} \tag{4.90}$$

The Hamiltonian (4.44) for the linear (or momentum) modes takes the following form:

$$H_{\text{lin}}(\vec{m}) = \frac{1}{\alpha'} p^\jmath \underbrace{\langle e_\jmath, e_\ell\rangle}_{G_{\jmath\ell}} p^\ell = \frac{1}{\alpha'} m_\jmath \Big(\big(G - \mathcal{F}_k^2\big)^{-1}\Big)^{\jmath\ell} m_\ell \quad k \in \{1,2\} \tag{4.91}$$

The expression in the big parentheses might be called the dual of the *open string metric* in a slight generalization of the open string metric introduced by Seiberg and Witten [83]. The metric introduced there does not include the $U(1)$ field strength. Eq. (4.91) is independent of k. Note however, that the $G_{\jmath\ell}$ is the metric of the lattice $\Gamma^{(d-p)}$, i.e. the lattice spanned by $e_\jmath$ with $(F_1+F_2)e_\jmath = 0$, and not of the full lattice Γ^d.

After solving the linear modes, we are still left with the zero modes, i.e. with the operators H^i, $\lambda_i \neq 1$. Their quantization is dictated by the invariance of the wavefunctions under translations by $2\pi\Gamma^p$. (The translations by $2\pi\Gamma^{p-d}$, are projected out in $\langle 2\pi\Gamma^{p-d}, (F_1+F_2)H\rangle$.) As the H^i (in contrast to the p^ℓ) do not commute, we can not get simultaneous Eigenfunctions of all H^i. However we can find a basis, such that (F_1+F_2) becomes block-diagonal (cf. (4.87)).[20] The respective pairs $\big(H_1^{(i)} e_i^{(1)}, H_2^{(i)} e_i^{(2)}\big)$ (no sums over i), then fulfill (cf. (4.87)):

$$e_i^{(1)} \frac{(F_1+F_2)}{\alpha'} e_j^{(2)} = f^{(i)} \delta_{ij} \tag{4.92}$$

We want the wavefunctions to be Eigenfunctions of one of the $H_j^{(i)}$. Without loss of generality we choose the set $H_1^{(i)}$. Under the translation $n^i \cdot 2\pi e_i^{(2)}$ an H_1-Eigenfunction with Eigenvalue H_1 acquires a phase:

$$\langle \Pi, n\cdot 2\pi e_i^{(2)}\rangle = 2\pi n^i \left(\frac{f^{(i)}}{2\pi} H_1^{(i)} + \frac{1}{\alpha'}\langle e_i^{(2)}, (G-\mathcal{F}_1)e_\ell\rangle p^\ell(\vec{m})\right) \tag{4.93}$$

Single valuedness of the wavefunction requires for the Eigenvalue of $H_1^{(i)}$:

$$H_1^{(i)}\big(l^{(i)}\big) = 2\pi \frac{l^{(i)}}{f^{(i)}} \qquad l^{(i)} \in \mathbb{Z} \tag{4.94}$$

[18] As Π commutes with the Hamiltonian, the Eigenfunctions of the magnetic translations T are Eigenfunctions of the Hamiltonian as well.

[19] By this we abandon the C_ℓ to form an ortho-normalized set of vectors.

[20] Note that the block-diagonal form in (4.87) does of course *not* imply that the (lattice) vectors belonging to the individual blocks are perpendicular w.r.t. G.

(as the second term on the r.h.s. in (4.93) vanishes). The inequivalent choices of $H_1^{(i)}$ are given by $l^{(i)} \in \{1 \dots f^{(i)}\}$. The wavefunctions are completely localized in the $H_1^{(i)}$ coordinate. Translations by $2\pi e_i^{(1)}$ map a wavefunction localized at $H_1^{(i)}$ to one localized at $H_1^{(i)} + 2\pi$. Therefore we do not need to forbid that the wavefunction picks up a phase under such a translation. Symmetrizing the wavefunction in the $e_i^{(1)}$ direction now means to add up all translated wavefunctions.

The algebra $\left[H_1^{(i)}, H_2^{(2)}\right] = \frac{i2\pi}{f^{(i)}}$ is not finitely represented by matrices, but admits the usual infinite-dimensional representation:

$$\phi\big(H_1^{(i)}(l^{(i)})\big) = \exp\big(- iH_1^{(i)}(l^{(i)})x^{(i)}\big), \qquad \hat{H}_1 \simeq i\partial_{x^{(i)}}\,,\ \hat{H}_2 \simeq \tfrac{2\pi}{f^{(i)}}x^{(i)} \tag{4.95}$$

The number of inequivalent states (i.e. the number that enters partition functions) is given by:[21]

$$n_0 = \prod_{j=1}^{p/2} f^{(j)} \qquad \left(= (-1)^{\lceil p/2 \rceil}\,\mathrm{Pf}\left(\frac{(F_1+F_2)_{ij}}{\alpha'}\right)\right) \tag{4.96}$$

From the representation of the Pfaffian we see that the multiplicity n_0 can be rewritten as:

$$n_0 = \mathrm{ch}_{p/2}\left((F_1+F_2)_{ij}\right) \tag{4.97}$$

If we are only interested in chiral degrees of freedom, which appear in superstring compactifications, we do not need to restrict to the invertible matrix $(F_1+F_2)_{ij}$. Possible bosonic momentum-modes imply vanishing chiral fermion number. Denoting the multiplicity of chiral states by ν_0 we can replace Chern character $\mathrm{ch}_{p/2}$ by the top Chern character $\mathrm{ch}_{d/2}$ in the following way:[22]

$$\nu_0 = \mathrm{ch}_{d/2}\left((F_1+F_2)\right) = \int_{T^d} \mathrm{ch}(F_1 \otimes F_2) = \int_{T^d} \mathrm{ch}\,F_1 \wedge \mathrm{ch}\,F_2 \tag{4.98}$$

This chiral multiplicity ν_0 is a special case of the *Atiyah-Singer index theorem* for twisted spin-complexes: [23]

$$\nu_0 = \int_{\mathcal{M}} \hat{\mathcal{A}}(\mathcal{R})\,\mathrm{ch}\,E = \int_{T^d} \mathrm{ch}\,F_1 \wedge \mathrm{ch}\,F_2 \tag{4.99}$$

Like in chapter 3, $\hat{\mathcal{A}}(\mathcal{R})$ denotes the A-roof genus of the tangent bundle $T\mathcal{M}$. $\mathrm{ch}\,E$ is the Chern character of the vector-bundle to which the gauge field F is the curvature two-form.

The multiplicities ν_0 can be interpreted as *Landau levels* that appear in the case of quantized point particles in a (constant) magnetic background field. In finite systems the degeneracy of Landau levels is finite as well. If spin is included (Pauli-equation), the Landau levels are split according to the spin $s = \pm 1/2$.

[21] $(F_1+F_2)_{ij}$ denotes the restriction of $(F_1+F_2)_{ij}$ to the subspace on which (F_1+F_2) is invertible.

[22] We assume the number of compactified dimension to be even.

[23] Cf. [97], p. 331-334 and [98], p. 420-424).

We want to conclude with a remark on the quantization condition for the F-fields (4.85). The torus group $2\pi\Gamma^d$ of the open string might be a proper subgroup of the closed string torus group (which we denote for clarity by $2\pi\tilde{\Gamma}^d$). In terms of the $\tilde{\Gamma}^d$ basis, $(F_1+F_2)/\alpha'$ might then be $\mathbb{Q}$-valued. The physical interpretation is as follows: The D-brane wraps the torus n-times ($n \in \mathbb{Z}$). Without F-field this would imply that additional massless modes appear, promoting the gauge symmetry from the $U(1)$ of the single D-brane to a $U(n)$-symmetry of n D-branes. Due to the F-field, even though the brane is multiply wrapped, no further massless modes show up, and the gauge symmetry stays a $U(1)$.

In what follows, we will make some comments on the further quantization procedure.

4.2.2 Hilbert-space, further quantization

So far we quantized oscillators as well as zero and linear (momentum) modes. If no electric components are present, and if the X^0 (time direction) has Neumann boundary condition, light-cone quantization might be applied. This procedure is then completely analogous to the quantization of branes at angles [89, 90, 91, 27, 99], except for the zero modes. There should exist a description for the vertex operators as well. For open strings with B-field (the general $\mathcal{F}$-field case works the same way with $F_1 = -F_2$) the vertex operators and their OPEs were given explicitly in [85, 83]. However the situation of $F_1 \neq F_2$, has not yet been *completely* explored to our knowledge.

If electric components are present, the conditions for applying light-cone quantization are no longer fulfilled.[24] One could then apply path-integral quantization.[25] The boundary conditions of the ghosts are unaffected by the $\mathcal{F}$-fields. The partition function is a product of the bosonic part and the ghost part. Roughly speaking, the ghost part is the inverse of the partition function of one complex boson with Neumann boundary conditions. From the Hamiltonian (4.45) and the Poisson bracket (4.66) we see that light-like Eigenvectors of V_2V_1 with Eigenvalues $\xi \neq \pm 1$ would imply the existence of particles with *complex* mass^2. This indicates surely an instability. Complex mass^2 are usually inserted into propagators of unstable particles. The instability encountered in the presence of electric fields was observed in string theory by Burgess [100], who considered the case of strings with independent charges on the end points but with identical F-field. For consistent quantization we will require for the Eigenvalue spectrum of V_2V_1:

$$|\lambda_i| = 1 \quad \forall \quad \text{Eigenvalues } \lambda_i \tag{4.100}$$

[24] This condition means that there exist at least one light-cone coordinate whose boundary condition is Neumann. Otherwise this coordinate can not be identified with a (transformed) world sheet time because the world sheet coordinate- and conformal transformations obey $\partial_\sigma\tilde{\tau}(\tau, \sigma = 0, \pi) = \partial_\sigma\tilde{\sigma}(\tau, \sigma = 0, \pi) = 0$ reflecting the fact that the boundary is mapped to itself.

[25] Even though the path integral formalism does not require the very special form of the mode expansion of light-like coordinates, as the light-cone formalism does, there are still some obstacles left. For example the path integral is only defined for euclidian space(-time). Even though one might argue that the minkowskian answer is obtained by (a second) Wick rotation, it is not clear if this method does not miss some points.

As the quantization of the remaining cases is completely analogous to the known cases cited above from now onwards, we will skip the rest of the procedure and turn to some aspect on non-commutativity that arise in the context of strings coupled to B and F fields.

4.3 The commutator $[X(\tau,\sigma), X(\tau,\sigma')]$

In what follows, we will calculate $[X(\tau,\sigma), X(\tau,\sigma')]$ and confirm by this the disk result of Seiberg and Witten for the one-loop case with general constant $\mathcal{F}$-fluxes on both branes. This is of course expected as the commutator usually only depends on local properties. For simplicity we restrict to the case without Dirichlet boundary conditions at some point. To make things simpler, we set τ to zero. However it is easily checked that the expressions that we calculate, i.e. the oscillator part and the part of the linear modes (the p_ℓ's) are independent of τ. The commutator part linear in the p_ℓ's cancels out and a part quadratic in the p_ℓ's does not exist, as the p_ℓ's commute (cf. (4.57)). The τ dependence of the oscillator part would drop out explicitly in (4.102) anyway since the commutator $\left\{a^j_{-n_j}, a^{-l}_{m_l}\right\}$ selects oscillators with opposite τ dependence in the exponential.

4.3.1 The commutator $[X(\tau,0), X(\tau,0)]$

We divide the Poisson bracket for the world sheet field $X(\tau,\sigma)$ into an oscillator, a zero-mode and a linear part. For $\sigma=\sigma'=0$ only the oscillator and zero mode part contribute:

$$\begin{aligned}\left\{X^\mu(\tau=0,\sigma), X^\nu(\tau=0,\sigma')\right\} = \\ \left\{X^\mu(\tau=0,\sigma)_{\mathrm{osc}}, X^\nu(\tau=0,\sigma')_{\mathrm{osc}}\right\} + \{H^\mu, H^\nu\}\end{aligned} \tag{4.101}$$

The oscillator part turns out to be:

$$\begin{aligned}&\left\{X^\mu(\tau=0,\sigma=0)_{\mathrm{osc}}, X^\nu(\tau=0,\sigma'=0)_{\mathrm{osc}}\right\} \\ &\quad= -\frac{\alpha'}{2}\sum_j\sum_l (G+V_1)C_j \sum'_{n_j\in\mathbb{Z}+\theta_j}\sum'_{m_l\in\mathbb{Z}+\theta_l}\frac{\left\{a^j_{-n_j}, a^{-l}_{m_l}\right\}}{n_j(-m_l)}C_l(G+V_1^T) \\ &\quad= -\frac{\alpha'}{2}\sum_j\sum_l (G+V_1)C_j \sum'_{n_j\in\mathbb{Z}+\theta_j}\frac{i}{n_{-j}}C_-j(G+V_1^T)\end{aligned} \tag{4.102}$$

The above expression is meaningless unless we do not regularize the divergent term $\sum'_{n_j\in\mathbb{Z}+\theta_j}\frac{i}{n_{-j}}$. We regularize by substituting this term:

$$\sum'_{n_j\in\mathbb{Z}+\theta_j}\frac{1}{n_{-j}} = \sum_{\substack{n\in\mathbb{Z}\\ n-\theta_j\neq 0}}\frac{1}{n-\theta_j} \to \left(\sum_{\substack{n\in\mathbb{Z}^*\\ n-\theta_j\neq 0}}\frac{1}{n-\theta_j}-\frac{1}{n}\right)-\frac{1}{\theta_j} = \pi\cot(-\pi\theta_j) \tag{4.103}$$

Using $\cot(-\pi\theta_j) = i(\lambda_j^{\frac{1}{2}} + \lambda_j^{-\frac{1}{2}})/(\lambda_j^{-\frac{1}{2}} - \lambda_j^{\frac{1}{2}})$ as well as orthogonality of the C_j, we can rewrite the regularized Poisson bracket as

$$\begin{aligned}&\big\{X^\mu(\tau,\sigma=0)_{\text{osc}}, X^\nu(\tau,\sigma'=0)_{\text{osc}}\big\}\\ &\quad = \frac{\pi\alpha'}{2}\sum_{\substack{j\,:\,\theta_j\neq 0\\ l\,:\,\theta_l\neq 0}}\Big((G+V_1)C_lC_{-l}(V_2V_1-1)^{-1}(V_2V_1+1)C_jC_{-j}(G+V_1^T)\Big)^{\mu\nu}\end{aligned} \tag{4.104}$$

The zero mode expression can be written as:

$$\begin{aligned}&\{H^\mu, H^\nu\} = \\ &\quad \frac{\alpha'}{2}\sum_{i,j}\sum_{a,b}\Big(\Big(e_i - d_a\big(N^{-1}\big)^T K^T\Big)\big(A_1^{-1}\big)\Big(e_j - K\big(N^{-1}d_b\big)\Big)\Big)^{\mu\nu}\\ &\qquad\qquad\qquad + \sum_{a,b}\big(d_aA_2d_b\big)^{\mu\nu}\end{aligned} \tag{4.105}$$

We will now restrict ourselves to the case without Dirichlet conditions. (The situation where the Dirichlet conditions are in directions which are perpendicular to *both* $\mathcal{F}$-fields is a trivial generalization of the case under consideration. Non-trivial is the calculation of the commutator if the Dirichlet conditions of one brane interfere with the directions in which the $\mathcal{F}$-field of the other brane points.) If all Eigenvalues λ are equal to one the oscillator part vanishes, as well as $(A_1)^{-1}$ and therefore only the A_2 term survives. A bit more complicated is the situation where $\lambda \neq 1$ for all Eigenvalues λ. However $\mathcal{F}_1 + \mathcal{F}_2$ is invertible as it stands in this case and the e_i can be an arbitrary basis of $\mathbb{R}^n$. A_2 vanishes as well as K and the resulting terms are easily summed up to yield the result stated by Seiberg and Witten in [83]. A generalization is to allow the Eigenvalues to be of both types. With the R-matrices (eq. (4.15)) that determine the Dirichlet direction now being the identity, the situation simplifies drastically. The Eigenvalue equation for λ_i is given by the vanishing of the determinant of the following matrix:

$$\begin{aligned}&(V_2V_1 - \lambda\,\mathrm{Id})^\mu{}_\nu = \\ &\quad\Big((G+\mathcal{F}_2)^{-1}\big[(1-\lambda)(G+\mathcal{F}_2G\mathcal{F}_1) - (1+\lambda)(\mathcal{F}_2+\mathcal{F}_1)\big](G+\mathcal{F}_1)^{-1}G\Big)^\mu{}_\nu\end{aligned} \tag{4.106}$$

We deduce that the $(G+\mathcal{F}_1)^{-1}GC_i$ or $(G-\mathcal{F}_2)^{-1}GC_i$ with eigenvalue $\lambda_i \neq 1$ can be chosen as the e_i in eq. (4.50) and the d_a from equation (4.53) (which are perpendicular to the e_i) are then given by $(G+\mathcal{F}_1)^{-1}GC_\ell = (G-\mathcal{F}_2)^{-1}GC_\ell$. We can read of both quantities $(V_2V_1 \pm \mathrm{Id})$ from (4.106). We will now rewrite the oscillator part in such a way, that the matrix A_1 appears on one side. This enables us to sum up this term with the $e_i(A_1)^{-1}e_j$ term from the zero mode part (4.105) as the "denominator" is then identical and isolated at one side.

We define the following two (orthogonal) projectors:

$$\mathbf{Pr}_1 \equiv \sum_{\lambda_i \neq 1} C_i C_{-i} \qquad\qquad \mathbf{Pr}_2 \equiv \sum_{\lambda_\ell = 1} C_\ell C_\ell \tag{4.107}$$

$$\Rightarrow \mathbf{Pr}_1 \mathbf{Pr}_2 = \mathbf{Pr}_2 \mathbf{Pr}_1 = 0 \qquad\qquad \mathbf{Pr}_1 + \mathbf{Pr}_2 = \mathrm{Id} \tag{4.108}$$

- We choose $e_i \equiv (G + \mathcal{F}_1)^{-1} C_i \; d_\ell \equiv (G \pm \mathcal{F}_{\frac{1}{2}})^{-1} C_\ell$.
- The matrices A_1 (eq. (4.50)), A_2, K and N (eq. (4.53)) now simplify to:[26]

$$\begin{aligned}
(A_1)_{ik} &= \frac{1}{4\pi} C_i (G - \mathcal{F}_1)^{-1} (\mathcal{F}_1 + \mathcal{F}_2)(G + \mathcal{F}_1)^{-1} C_k && \lambda_i,\, \lambda_k \neq 1 \\
(K)_{i\ell} &= \frac{1}{2} C_i (G - \mathcal{F}_1)^{-1} (G + \mathcal{F}_2) C_\ell \quad (= 0) && \lambda_i \neq 1,\, \lambda_\ell = 1 \\
(N)_{\jmath\ell} &= \frac{1}{2} C_\jmath G C_\ell && \lambda_\jmath = \lambda_\ell = 1 \\
(A_2)_{\jmath\ell} &= \pi C_\ell \mathcal{F}_2 C_\jmath
\end{aligned} \tag{4.109}$$

and we note that

$$\begin{aligned}
&\sum_{\lambda_m \neq 1} C_{-k}((C_{-l}(V_2 V_2 - 1) C_j)^{-1})^{-km} C_m C_{-m} (V_2 V_2 + 1) C_n = \\
&\frac{-1}{4\pi} C_{-k} \big((A_1)^{-1}\big)^{k,-m} \overbrace{C_m (G - \mathcal{F}_1)^{-1} (G + \mathcal{F}_2) \mathbf{Pr}_1 (G + \mathcal{F}_2)^{-1}}^{= C_m (G - \mathcal{F}_1)^{-1}} (G + \mathcal{F}_2 \mathcal{F}_1) C_n
\end{aligned} \tag{4.110}$$

Using this result we sum up the oscillator contribution (4.104) with the term $e_i (A_1)^{-1} e_j$ from the zero mode contribution (4.105):

$$\begin{aligned}
&\Big(\big\{X^\mu(\tau = 0, \sigma)_{\mathrm{osc}}, X^\nu(\tau = 0, \sigma')_{\mathrm{osc}}\big\} + \frac{\alpha'}{2} e_i \big(A_1^{-1}\big)^{ij} e_j\Big)^{\mu\nu} \\
&= \frac{\alpha'}{2} \Big(-(G + \mathcal{F}_1) C_i \big((A_1)^{-1}\big)^{i,j} \\
&\qquad C_{-j} \big((G - \mathcal{F}_1)^{-1} (G + \mathcal{F}_2 \mathcal{F}_1)(G + \mathcal{F}_1)^{-1} \mathbf{Pr}_1 (G - \mathcal{F}_1)^{-1} \\
&\qquad\qquad + (G - \mathcal{F}_1)^{-1}\big)\Big)^{\mu\nu}
\end{aligned} \tag{4.111}$$

As

$$\begin{aligned}
&C_{-j} (G - \mathcal{F}_1)^{-1} \\
&\quad = C_{-j} (G - \mathcal{F}_1)^{-1} (G - \mathcal{F}_1)(G + \mathcal{F}_1)(G + \mathcal{F}_1)^{-1} (\mathbf{Pr}_1 + \mathbf{Pr}_2)(G - \mathcal{F}_1)^{-1} \\
&\qquad = C_{-j} (G - \mathcal{F}_1)^{-1} (G - \mathcal{F}_1 \mathcal{F}_1)(G + \mathcal{F}_1)^{-1} \mathbf{Pr}_1 (G - \mathcal{F}_1)^{-1}
\end{aligned} \tag{4.112}$$

[26] With the help of relation (4.123) which will be derived in this section, we can even show, that $K_{i\ell}$ vanishes completely.

for $\lambda_j \neq 1$ (as it is the case), the oscillator part simplifies to:

$$\begin{aligned}
&\Big(\big\{X^\mu(\tau,0)_{\mathrm{osc}}, X^\nu(\tau,0)_{\mathrm{osc}}\big\} + \frac{\alpha'}{2} e_i (A_1)^{-1} e_j\Big)^{\mu\nu} \\
&\quad = -2\pi\alpha' \Big((G+\mathcal{F}_1)^{-1} C_i \big((A_1)^{-1}\big)^{i,j} (A_1)_{i,j} \mathbf{Pr}_1 \mathcal{F}_1 \mathbf{Pr}_1 (G-\mathcal{F}_1)^{-1}\Big)^{\mu\nu} \\
&\quad\quad = -2\pi\alpha' \Big((G+\mathcal{F}_1)^{-1} \mathbf{Pr}_1 \mathcal{F}_1 \mathbf{Pr}_1 (G-\mathcal{F}_1)^{-1}\Big)^{\mu\nu}
\end{aligned} \tag{4.113}$$

Thus we already obtained a first part of the Poisson bracket. Before we continue to calculate other parts of the zero-mode contribution, we will derive some extremely useful relations. As annotated, we will see that with our choice of the d_ℓ, the matrix K cf. (4.109) vanishes. We read of from equation (4.106) that

$$\begin{aligned}
\mathbf{Pr_I} &\equiv (G+\mathcal{F}_1)^{-1}\mathbf{Pr}_2(G+\mathcal{F}_1) \quad \text{and} \quad & \mathbf{Pr_{II}} &\equiv (G-\mathcal{F}_2)^{-1}\mathbf{Pr}_2(G-\mathcal{F}_2) \\
&= (G-\mathcal{F}_2)^{-1}\mathbf{Pr}_2(G+\mathcal{F}_1) & &= (G+\mathcal{F}_1)^{-1}\mathbf{Pr}_2(G-\mathcal{F}_2)
\end{aligned} \tag{4.114}$$

are projectors. As $\mathbf{Pr_{III}} \equiv \mathbf{Pr_I}\mathbf{Pr_{II}}$ turns out to be a projector, too, $\mathbf{Pr_I}$ and $\mathbf{Pr_{II}}$ commute:

$$\mathbf{Pr_I}\mathbf{Pr_{II}} = \mathbf{Pr_{II}}\mathbf{Pr_I} \tag{4.115}$$

$$\Rightarrow (G+\mathcal{F}_1)^{-1}\mathbf{Pr}_2(G-\mathcal{F}_2) = \Big(G \mp \mathcal{F}_{\substack{2\\1}}\Big)^{-1}\mathbf{Pr}_2(G+\mathcal{F}_1) \tag{4.116}$$

By multiplying the last result with the (invertible) matrix $(G+\mathcal{F}_1)$ we note that:

$$\mathbf{Pr}_2\mathcal{F}_1 = -\mathbf{Pr}_2\mathcal{F}_2 \qquad \text{and} \qquad \mathcal{F}_1\mathbf{Pr}_2 = -\mathcal{F}_2\mathbf{Pr}_2 \tag{4.117}$$

with the right equality being the transpose of the left. We also observe now that $\mathbf{Pr_I} = \mathbf{Pr_{II}}$. As the projectors

$$\mathbf{Pr_A} \equiv (G+\mathcal{F}_1)^{-1}\mathbf{Pr}_1(G+\mathcal{F}_1) \quad \text{and} \quad \mathbf{Pr_B} \equiv (G-\mathcal{F}_2)^{-1}\mathbf{Pr}_1(G-\mathcal{F}_2) \tag{4.118}$$

fulfill:

$$\mathbf{Pr_A}\mathbf{Pr_I} = 0 \qquad \mathbf{Pr_B}\mathbf{Pr_I} = 0 \qquad \mathbf{Pr_A} + \mathbf{Pr_I} = \mathbf{Pr_B} + \mathbf{Pr_I} = \mathrm{Id} \tag{4.119}$$

we conclude:

$$\mathbf{Pr_A} = \mathbf{Pr_B} \tag{4.120}$$

Rewriting $C_j(G-\mathcal{F}_1)^{-1}(G+\mathcal{F}_2)C_\ell = 0$ $(\lambda_i \neq 1, \lambda_\ell = 1)$ (cf. eq. (4.117)) we get:

$$\underbrace{C_i(G-\mathcal{F}_1)^{-1}(G-\mathcal{F}_1)C_\ell}_{=0} + C_i(G-\mathcal{F}_1)^{-1}(\mathcal{F}_1+\mathcal{F}_2)C_\ell = 0 \tag{4.121}$$

Inserting into this expression the identity $(G+\mathcal{F}_1)^{-1}(\mathbf{Pr}_1+\mathbf{Pr}_2)(G+\mathcal{F}_1)$ we obtain (the term $\propto (\mathcal{F}_1+\mathcal{F}_2)(G+\mathcal{F}_1)^{-1}(\mathbf{Pr}_2)$ vanishes):

$$C_i(G-\mathcal{F}_1)^{-1}(\mathcal{F}_1+\mathcal{F}_2)(G+\mathcal{F}_1)^{-1}C_{-k}C_k(G+\mathcal{F}_1)C_\ell = 0 \quad \forall i \tag{4.122}$$

As $(A_1)_{i,-k} = C_i(G-\mathcal{F}_1)^{-1}(\mathcal{F}_1+\mathcal{F}_2)(G+\mathcal{F}_1)^{-1}C_{-k}$ is a non-singular matrix, we finally arrive at:

$$C_k\mathcal{F}_1C_\jmath = C_k\mathcal{F}_2C_\ell = 0 \qquad \lambda_k \neq 1,\, \lambda_\ell = 1 \tag{4.123}$$

As a consequence K in (4.109) and (4.105) vanish as well. The remaining term in (4.105) is $\propto \sum_{\jmath,\ell} d_\jmath N^{-1}A_2^{-1}N^{-1}d_\ell$. It can be rewritten by (4.117):

$$\sum_{\jmath,\ell} d_\jmath N^{-1}A_2N^{-1}d_\ell = -2\pi\alpha'(G+\mathcal{F}_1)^{-1}\mathbf{Pr}_2\mathcal{F}_1\mathbf{Pr}_2(G+\mathcal{F}_1)^{-1} \tag{4.124}$$

The last line in (4.124) equals $-(G-\mathcal{F}_2)^{-1}\mathcal{F}_2\mathbf{Pr}_2(G+\mathcal{F}_2)^{-1}$ and cancels therefore the A_2 term in (4.105). We finally arrive at:

$$[X(\tau,0),X(\tau,0)] = i2\pi\alpha'\frac{\mathcal{F}_1}{G-\mathcal{F}_1^2} \tag{4.125}$$

4.3.2 The commutator $[X(\tau,\pi),X(\tau,\pi)]$

In this section we will calculate the commutator at the other end of an open string stretching between two branes. The situation is very similar compared to the $\sigma=0$ case. The zero mode part $\{H^\mu,H^\nu\}$ is unchanged and the linear (or: momentum modes) do not contribute for the following reason: The Poisson bracket for the zero- and momentum-modes (4.57) does not contain a term $\propto \{p^\imath,p^\jmath\}$. The τ-linear term that is proportional to $\{H^\imath,p^\jmath\}$ is symmetric, and does not contribute to the commutator (c.f. (4.48) and (4.57)). We read off from (4.57) that there are no terms $\{H^i,p^\jmath\}$ with H^i coupling to $(G+\mathcal{F}_1)^{-1}C_i$ (C_i an Eigenvector with Eigenvalue $\lambda_i\neq 1$). However a new term proportional to $\sigma\{H^\jmath,p^\ell\}$ (and its transpose) contributes now (c.f. (4.48)):

$$\begin{aligned}&2\pi\big((G+\mathcal{F}_1)C_\jmath\{H^\jmath,p^\ell\}C_\ell(-\mathcal{F}_1)(G-\mathcal{F}_1)\big)^{\mu\nu}\\ &\qquad\qquad = +2\pi\alpha'\big((G+\mathcal{F}_1)\mathbf{Pr}_2\mathcal{F}_1(G-\mathcal{F}_1)\big)^{\mu\nu}\end{aligned} \tag{4.126}$$

This term cancels (4.124). However there is the "transposed" term as well which gives the final momentum contribution to the commutator:

$$\begin{aligned}&2\pi\big((G+\mathcal{F}_1)\mathcal{F}_1C_\jmath\{p^\jmath,H^\ell\}C_\ell(G-\mathcal{F}_1)\big)^{\mu\nu}\\ &\qquad\qquad = -2\pi\alpha'\big((G-\mathcal{F}_2)\mathbf{Pr}_2\mathcal{F}_2\mathbf{Pr}_2(G+\mathcal{F}_2)\big)^{\mu\nu}\end{aligned} \tag{4.127}$$

The Poisson-bracket of the oscillator part changes however slightly: Due to terms of the type

$$(Ge^{in_i(\tau+\sigma)}+V_1e^{in_i(\tau-\sigma)})C_i \tag{4.128}$$

appearing in the mode expansion of $X(\tau,\sigma)$ (eq. (4.36)) the $(G+V_1)$ term in (4.102) changes to

$$\begin{aligned}\big(Ge^{in_j\pi}+V_1e^{-in_j(\pi)}\big)C_j &= (-1)^n\Big(\lambda_j^{-\frac{1}{2}}(\lambda_jGC_j+\underbrace{V_1C_j}_{\lambda_jV_2^{-1}C_j})\Big)\\ &= (-1)^n\lambda_j^{\frac{1}{2}}(G+V_2^{-1})C_j = (-1)^n\lambda_j^{\frac{1}{2}}(G-\mathcal{F}_2)^{-1}C_j\end{aligned} \tag{4.129}$$

In the same way $C_{-j}(G+V_1^T)$ changes to

$$C_{-j}\big(Ge^{-in_j\pi}+V_1^Te^{+in_j(\pi)}\big) \\ =(-1)^n\lambda_{-j}^{\frac{1}{2}}C_{-j}\Big(G+\big(V_2^{-1}\big)^T\Big)=(-1)^n\lambda_{-j}^{\frac{1}{2}}C_{-j}\big(G+\mathcal{F}_2\big)^{-1} \quad (4.130)$$

As $\lambda_j^{\frac{1}{2}}\lambda_{-j}^{\frac{1}{2}}=1$ this factor cancels out and the analog of (4.104) becomes:

$$\big\{X^\mu(\tau,\sigma=\pi)_{\text{osc}},X^\nu(\tau,\sigma'=\pi)_{\text{osc}}\big\} \\ =\frac{\pi\alpha'}{2}\sum_{\substack{j\,:\,\theta_j\neq 0\\ l\,:\,\theta_l\neq 0}}\Big((G+V_2)C_lC_{-l}(V_2V_1+1)(V_2V_1-1)^{-1}C_jC_{-j}(G+V_2^T)\Big)^{\mu\nu} \\ (4.131)$$

where we interchanged the order of (V_2V_1+1) and $(V_2V_1-1)^{-1}$. We express the above equation now partially in terms of $\mathcal{F}_1$ and $\mathcal{F}_2$:

$$\big\{X^\mu(\tau,\sigma=\pi)_{\text{osc}},X^\nu(\tau,\sigma'=\pi)_{\text{osc}}\big\} \\ =-2\pi\alpha'\sum_{\substack{i\,:\,\theta_i\neq 0\\ j\,:\,\theta_j\neq 0}}\Big((G-\mathcal{F}_2)^{-1}\mathbf{Pr}_1(G-\mathcal{F}_2)^{-1}(G+\mathcal{F}_2\mathcal{F}_1)(G+\mathcal{F}_1)^{-1}C_i \\ (V_2V_1-1)^{-1}_{-i,j}C_-j(G+\mathcal{F}_2)^{-1}\Big)^{\mu\nu} \quad (4.132)$$

Next we will rewrite the term $e_i\big(A_1^{-1}\big)^{ij}e_j$ that stems from the zero mode part:

$$4\pi(A_1)_{-i,k}=C_{-i}(G-\mathcal{F}_1)^{-1}(G+\mathcal{F}_2)(G+\mathcal{F}_2)^{-1}(\mathcal{F}_1+\mathcal{F}_2)(G+\mathcal{F}_1)^{-1}C_{-k} \\ =\sum_{j\,:\,\theta_j\neq 0}C_{-i}(G-\mathcal{F}_1)^{-1}(G+\mathcal{F}_2)C_{-j}C_j(G+\mathcal{F}_2)^{-1}(\mathcal{F}_1+\mathcal{F}_2)(G+\mathcal{F}_1)^{-1}C_{-k} \\ (4.133)$$

With the help of (4.133) we get:

$$e_i\big((A_1)^{-1}\big)^{i,j}e_j=4\pi(G+\mathcal{F}_1)^{-1}C_i(C_i\big(V_2V_1-1)C_j)^{-1}\big)_{-i,j}C_{-j}(G+\mathcal{F}_2)^{-1} \\ (4.134)$$

We can now add up the above expression with the oscillator part (4.132). Furthermore we multiply (4.134) by $\mathrm{Id}=(G-\mathcal{F}_2)^{-1}(\mathbf{Pr}_1+\mathbf{Pr}_2)(G+\mathcal{F}_2)^{-1}(G-\mathcal{F}_2^2)$:[27]

$$\Big(\big\{X^\mu(\tau,\pi)_{\text{osc}},X^\nu(\tau,\pi)_{\text{osc}}\big\}+\frac{\alpha'}{2}e_i\big(A_1^{-1}\big)^{ij}e_j\Big)^{\mu\nu} \\ =-2\pi\alpha'\Big((G-\mathcal{F}_2)^{-1}\mathbf{Pr}_1(G+\mathcal{F}_2)^{-1}\mathcal{F}_2(\mathcal{F}_1+\mathcal{F}_2)(G+\mathcal{F}_1)^{-1}C_i \\ (C_i\big(V_2V_1-1)C_j)^{-1}\big)_{-i,j}C_{-j}(G+\mathcal{F}_2)^{-1}\Big)^{\mu\nu} \\ =-2\pi\alpha'\Big((G-\mathcal{F}_2)^{-1}\mathbf{Pr}_1\mathcal{F}_2\mathbf{Pr}_1(G+\mathcal{F}_2)^{-1}\Big)^{\mu\nu} \quad (4.135)$$

[27]Only the $\mathbf{Pr}_1$ contributes after multiplying with $e_i(A_1)^{-1}e_j$.

Finally we rewrite the commutator:

$$[X(\tau,\pi),X(\tau,\pi)]=i2\pi\alpha'\frac{\mathcal{F}_2}{G-\mathcal{F}_2^2} \tag{4.136}$$

Besides this result, we found that in the basis (C_i, C_ℓ) both $\mathcal{F}$-fields are of the form:

$$\left(\frac{C_i}{C_\jmath}\right)\mathcal{F}_1\left(\frac{C_k}{C_\ell}\right)=\left(\begin{array}{c|c} C_i(\mathcal{F}_1)C_k & 0 \\ \hline 0 & C_\jmath(\mathcal{F}_1)C_\ell \end{array}\right) \tag{4.137}$$

$$\left(\frac{C_i}{C_\jmath}\right)\mathcal{F}_2\left(\frac{C_k}{C_\ell}\right)=\left(\begin{array}{c|c} C_i(\mathcal{F}_2)C_k & 0 \\ \hline 0 & -C_\jmath(\mathcal{F}_1)C_\ell \end{array}\right) \tag{4.138}$$

The lower block can be brought again to block-diagonal form with 2×2 blocks. The upper blocks can of course not be brought to such a form simultaneously. Summarizing, we reproduced and generalized the commutator relation presented in the cited papers to the one-loop case with constant, but otherwise completely unrestricted[28] electro-magnetic NS-$U(1)$ field strengths F_1 and F_2 at both boundaries, in addition to a constant NSNS two-form flux B.

4.4 Space-time supersymmetry of open strings in constant backgrounds

Supersymmetry in string-theory (as well as in field theory) implies in general vanishing of the (complete) partition function. At one loop-level this is due to Bose-Fermi degeneracy. For simplicity we restrict ourselves to the case without Eigenvectors C_λ belonging to an Eigenvalue $|\lambda|\neq 1$ in this section. We assume furthermore, that the we have Neumann boundary conditions in at least four (or six) space-time dimensions, while the remaining six (or four) dimensions of the superstring might have both Dirichlet and mixed boundary conditions.[29] By the identity (A.12) (and for orbifolds: (A.13)) in appendix A.1.2 we see that a necessary condition for a vanishing cylinder partition function A^{ij} in a sector given by the boundary conditions V_iV_j is a condition on its Eigenvalues:

$$\begin{array}{ll} d=4 & d=6 \\ 0=\theta_1+\theta_2 & 0=\theta_1+\theta_2+\theta_3 \end{array} \tag{4.139}$$
$$\text{with Eigenvalues } \{\lambda_i,\bar\lambda_i\} \text{ and } \lambda_i=e^{i2\pi\theta_i}$$

By an exchange $\lambda_i\to\bar\lambda_i$ the equations (4.139) will no longer be fulfilled. The meaning of (4.139) is clear: the rotation V_iV_j is not only $\in O(n)$ but even contained in the much smaller group $SU(n/2)$ with $n=4$ (or $n=6$). If two branes should be supersymmetric w.r.t. each other, it is sufficient,[30] that $V_2V_1\in SU(n/2)$. With several D-branes we will distinguish two cases:

1. Given a set of boundary conditions specified by matrices $\{V_i\}$ some products V_jV_i do not allow to choose Eigenvalues $\lambda_k^{(j,i)}$ s.th. $\sum_{k=1}^{n/2}\theta_k$ vanishes

[28] Neglecting some degenerate cases.
[29] By "mixed" we mean that an $\mathcal{F}$ field can be included.
[30] Not quite, but we will come to that point soon.

(with θ_k in the sum belonging to Eigenvectors that are not obtained by complex conjugation). This means that supersymmetry is broken in this sector, and $V_jV_i \notin SU(n/2)$.

2. For all sectors V_jV_i there exists a set of Eigenvalues$\{\lambda_i = \exp(i2\pi\theta_i)|\ \lambda_i \neq \bar{\lambda}_j\}$, s.th. $\sum_{i=1}^{n/2}\theta_i$ vanishes. This implies that $V_jV_i \in SU_{j,i}(n/2)$.[31]

While it is clear that in the first case all supersymmetry is broken by the D-branes, the second case is more subtle. Even though in this case all products V_jV_i lie in an $SU(n/2)$, the embedding of the individual $SU(n/2)$ into $SO(n)$ might differ for different choices of different pairs (i,j). There is also no warranty that the bulk supersymmetry will be preserved. In general, supersymmetry will only be preserved iff there exists a spinor ϵ_L (of definite chirality, namely the one of the left-moving closed string sector), such that the following spinor equation holds:[32]

$$\epsilon_L = \Sigma(V_iV_j)\epsilon_L \qquad \forall i,j \tag{4.140}$$

This can be seen as follows: Writing the left-moving supercharge as: $Q_L = \sum_\alpha(\epsilon_L)_\alpha Q_{L,\alpha}$ and the right moving one as $Q_R = \sum_\alpha(\epsilon_R)_\alpha Q_{R,\alpha}$ the combination $Q_L + Q_R$ will be preserved by the D-brane with boundary condition i, iff:

$$\epsilon_R = \Sigma(V_j)\epsilon_L \tag{4.141}$$

So far we omitted spinor indices in order to leave the possibility to consider either type IIA (with opposite chiralities for left and right movers) or type IIB (with identical chiralities for left and right movers). Therefore (4.140) and (4.141) should be understood with correct indices (i.e. projections). If for example eq. (4.141) has no solutions for given chiralities of ϵ_L and ϵ_R, this means that the single brane breaks already all bulk supersymmetry. On the other hand, (4.141) can admit several solutions. Combining (4.141) for two different boundary conditions i and j implies equation (4.140).[33]

Our analysis is very close to the one described in [8], chap. 13 and [101].

4.4.1 Closed form of the Eigenvalues λ_i in $d = 4$, 6.

The (truncated[34]) characteristic polynomial for an O(n) rotation is very restricted, as its Eigenvalues are forced to have modulus one. Furthermore we assume to have even an $SO(n)$ rotation, which is the case, if the dimension of the brane is the correct one for the theory under consideration (type IIA or type

[31] The subscript j,i denotes that the $SU_{j,i}$ belongs to the sector given by V_jV_i. Different sectors might lead to different embeddings of the associated $SU(n/2)$ into $SO(n)$.

[32] Σ denotes the spinor representation.

[33] Deriving eq. (4.27),p. 92 we implicitly assumed, that the $i = 2$ boundary has F-flux $-F_2$ (cf. (4.23), p. 91). This explains why in (4.140) instead of $V_i^{-1}V_j$ we have to take: V_iV_j.

[34] By truncated we mean that we divided by $(1-\lambda)^{(10-d)}$ in order to remove the $\lambda = 1$ Eigenvalues of the $10-d$ dimensional space-time.

IIB). From *Vietas' theorem* on roots we conclude that the characteristic polynomial $\chi(\lambda)$ is symmetric and takes the following form in $d = 4$ dimensions:[35]

$$\chi_4(\lambda) \propto a\lambda^4 + b\lambda^3 + c\lambda^2 + b\lambda + a \tag{4.142}$$

In four dimensions all three coefficients a, b, c can be extracted by inserting $\lambda = 0$ and $\lambda = \pm 1$ into $\chi_4(\lambda)$:

$$a = \chi_4(0), \quad b = \frac{1}{4}\big(\chi_4(1) - \chi_4(-1)\big), \quad c = \frac{1}{2}\big(\chi_4(1) + \chi_4(-1)\big) - 2\chi_4(0) \tag{4.143}$$

Dividing (4.142) by λ^2 and applying a basic identity for $\cos n\phi$, transforms this equation into a second order polynomial in $\cos \pi\theta$.[36] The two solutions for $\cos(\pi\theta)$ are given by:

$$\cos(\pi\theta_{1/2}) = \frac{-b \pm \sqrt{b^2 + 8\,a^2 - 4\,ac}}{4\,a} \tag{4.144}$$

In $d = 6$ two additional terms show up:

$$\chi_6(\lambda) \propto a\lambda^6 + b\lambda^5 + c\lambda^4 + d\lambda^3 + c\lambda^2 + b\lambda + a \tag{4.145}$$

To extract all four coefficients in χ_6 one has to insert $\lambda = i$ as well. Doing so we get:

$$\begin{aligned} a &= \chi_6(0) & b &= \frac{\chi_6(1) - \chi_6(-1)}{2} - i\frac{\chi_6(i)}{4} \\ c &= \frac{\chi_6(1) + \chi_6(-1)}{4} - \chi_6(0) & d &= \frac{\chi_6(1) - \chi_6(-1)}{4} + i\frac{\chi_6(i)}{2} \end{aligned} \tag{4.146}$$

By dividing (4.145) by λ^3, the Eigenvalue equation can be rewritten as a third order polynomial in $x = \cos \pi\theta$:

$$x^3 + \underbrace{\frac{b}{2a}}_{\equiv r} x^2 + \underbrace{\frac{c - 3\,a}{4\,a}}_{\equiv s} x + \underbrace{\frac{d - 2\,b}{8\,a}}_{\equiv t} = 0 \tag{4.147}$$

The above equation must admit three real solutions (possibly some of them coincident) if we express the coefficients via (4.146). We can now apply a well known formula to solve this equation in closed form. However expressed in terms of a, b, c, d these solutions are quite lengthy. The explicit solutions blow up even more if we express the result in terms of $\chi_6(0), \chi_6(\pm 1)$ and $\chi_6(i)$. Therefore we forego without printing the three solutions. To make numerical calculations simpler, one can also multiply the characteristic polynomial by $\det(G + \mathcal{F}_2) \cdot \det(G + \mathcal{F}_1)$ thereby saving two matrix inversions.

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[35] The characteristic polynomial is the determinant of $(V_2 V_1 - \lambda\,\mathrm{Id})$, cf. eq. (4.19), p. 91 and eq. (4.106), p. 107.

[36] Remember that λ is related to θ by $\lambda = \exp(-i2\pi\theta)$.

It is very simple to impose the *necessary* supersymmetry condition (4.139) to the $d = 4$ dimensional case. This leads to one equation: (4.139) implies that the char. polynomial $\chi_4(\lambda)$ can be written as:

$$\chi_4(\lambda) = a(\lambda_1 - \lambda)^2(\bar{\lambda}_1 - \lambda)^2 \tag{4.148}$$

As two solutions coincide in that case, the root in (4.144) must vanish:

$$b^2 + 8a^4 = 4ac \tag{4.149}$$

It is an easy exercise to check that this condition is indeed fulfilled in the simple cases of tilted D-branes extending in two directions of the four dimensional space if the sum of their (oriented) angles vanishes. The same is true for the case of self-dual and anti-self dual field strengths.

We conclude this section with a reference to the publications [102] and [103]. In both publications the condition for membranes and D-branes to preserve supersymmetry were investigated. The approach pursued there is to look at the low energy effective action of the D-brane (membrane). It also captures the case of curved branes and non-constant NS F-fields. The classification of the different branes with their supersymmetry conditions is quite involved, and one has to consider branes of different dimensions separately. While [102] is restricted to vanishing background fields F and B, [103] considers the case that these fields are switched on. In [103] it was also derived that supersymmetric D-branes of real dimension three embedded in six-dimensions are not allowed to have an $\mathcal{F}_{\|}$-field living on it. This is not true for for supersymmetric D-branes of real dimension two embedded in a complex two-dimensional space. We will make some further comments in section 6.5.1 on configurations with $\mathcal{F}_{\|} = 0$ and D-branes with half the dimension of the embedding space in which case the supersymmetry condition reduces to a so called *special Lagrangian submanifold* (short *sLag*) condition.

Concluding remarks

In this chapter we quantized the open string for linear boundary conditions, that arise if two constant $U(1)$ field strengths couple to the string's boundaries. This quantization is new compared to what has been published in literature in several respects:

- Both field strengths are independent from each other.
- We can include Dirichlet conditions as well as so called *mixed boundary conditions*.
- We have in addition to non equal NS field strengths F_1 and F_2 the NSNS two-form potential B included.
- A quantization for the zero and momentum modes in arbitrary toroidal compactifications is derived from first principles (for the case without Dirichlet boundary conditions).

Some of the results are employed in chapter 5 and 6. However the method developed is applicable for far more general toroidal orbifold- and orientifold-constructions.

Chapter 5

Asymmetric Orientifolds

5.1 Introduction

As is known since the work of Connes, Douglas and Schwarz [104], matrix theory compactifications on tori with background three-form flux lead to non-commutative geometry. Starting with the early work [105] one has subsequently realized that open strings moving in backgrounds with non-zero two-form flux or non-zero gauge fields have mixed boundary conditions leading to a non-commutative geometry on the boundary of the string world-sheet [106, 107, 108, 85, 87, 83, 109, 110, 111, 112, 84, 86, 113]. We calculated the commutator at the string boundary in the last chapter for the one loop case and found agreement with the literature.

As pointed out in [83], also the effective theory on the D-branes becomes a non-commutative Yang-Mills theory.

We know from the discovery of D-branes, that Dirichlet branes made their first appearance by studying the realization of T-duality on a circle in the open string sector [114]. For instance, starting with a D9 brane, the application of T-duality leads to a D8-brane where the ninth direction changes from a Neumann boundary condition to a Dirichlet boundary condition. Thus, one may pose the question how D-branes with mixed Neumann-Dirichlet boundary conditions fit into this picture. Does there exist a transformation relating pure Dirichlet or Neumann boundary conditions to mixed Neumann-Dirichlet boundary conditions?[1] At first sight unrelated, there exists the so far unresolved problem of what the D-brane content of asymmetric orbifolds is. The simplest asymmetric orbifold is defined by modding out by T-duality itself, which is indeed a symmetry as long as one chooses the circle at the self-dual radius. Thus, as was argued in [115] and applied to Type I compactifications in [116], in this special case D9- and D5-branes are identified under the asymmetric orbifold action. However, the general T-duality group for compactifications on higher dimensional tori contains more general asymmetric operations. For instance,

[1]Even though we answered this question already partially in section 3.3 (p. 64), the discussion presented in this chapter is very close to our original paper [1]. We think it is illustrative to adopt main parts of the paper, while being even more specific in some points. Thereby most parts of this chapter can be read independently from the previous chapter.

the root lattice of $SU(3)$ allows an asymmetric $\widehat{\mathbb{Z}}_3$ action.[2] We made contact with this symmetry already in section 2.2.2.1 of the introductory chapter on orbifolds. In section 2.4 (p. 47) we considered the orbifold $T^4/(\mathbb{Z}_3^{\mathrm{L}} \times \mathbb{Z}_3^{\mathrm{R}})$.

The closed string sector can very well live with such non-geometric symmetries [41] but what about the open string sector? Since all Type II string theories contain open strings in the non-perturbative D-brane sector, in order for asymmetric orbifolds to be non-perturbatively consistent, one has to find a realization of such non-geometric symmetries in the open string sector, as well. Thus, the question arises what the image of a D9-brane under an asymmetric $\widehat{\mathbb{Z}}_N$ action is.

The third motivation for the investigation performed in this chapter is due to recently introduced orientifolds with D-branes at angles [27, 56, 117, 57, 118]. We investigated orientifold models for which the world-sheet parity transformation, Ω, is combined with a complex conjugation, $\bar{\sigma}$, of the compact coordinates. After dividing by a further left-right symmetric $\mathbb{Z}_N$ space-time symmetry the cancellation of tadpoles required the introduction of so-called twisted open string sectors. These sectors were realized by open strings stretching between D-branes intersecting at non-trivial angles. As was pointed out in [56], these models are related to ordinary Ω orientifolds by T-duality. However, under this T-duality the former left-right symmetric $\mathbb{Z}_N$ action is turned into an asymmetric $\widehat{\mathbb{Z}}_N$ action in the dual model. Thus, we are led to the problem of describing asymmetric orientifolds in a D-brane language. Note, that using pure conformal field theory methods asymmetric orientifolds were discussed recently in [53].

In this chapter, we study the three conceptually important problems mentioned above, for simplicity, in the case of compactifications on direct products of two-dimensional tori. It turns out that all three problems are deeply related. The upshot is that asymmetric rotations turn Neumann boundary conditions into mixed Neumann-Dirichlet boundary conditions. This statement is the solution to the first problem and allows us to rederive the non-commutative geometry arising on D-branes with background gauge fields simply by applying asymmetric rotations to ordinary D-branes. The solution to the second problem is that asymmetric orbifolds necessarily contain open strings with mixed boundary conditions. In other words: D-branes manage to incorporate left-right asymmetric symmetries by turning on background gauge fluxes, which renders their world-volume geometry non-commutative[3]. Gauging the left-right asymmetric symmetry can then lead to an identification of commutative and non-commutative geometries. In this sense asymmetric Type II orbifolds are deeply related to non-commutative geometry. Apparently, the same holds for asymmetric orientifolds, orbifolds of Type I. Via T-duality the whole plethora of $\Omega\bar{\sigma}$ orientifold models of [56, 117, 57] is translated into a set of asymmetric orientifolds with D-branes of different commutative and non-commutative types in the background. We will further present a D-brane interpretation of some of the non-geometric models studied in [53] and generalizations thereof.

[2] A left-right asymmetric $\mathbb{Z}_N$ symmetry is denoted by $\widehat{\mathbb{Z}}_N$ (cf. section 2.2.2.1).

[3] There is an exception for asymmetric orbifolds with $\widehat{\mathbb{Z}}_2$-action. The orientifold in [116] of such a model is consistent with D9- and D5-branes without any fluxes.

In section 5.2 we describe a special class of asymmetric orbifolds on T^2. Employing T-duality we first determine the tori allowing an asymmetric $\widehat{\mathbb{Z}}_N$ action, where we discuss the $\widehat{\mathbb{Z}}_3$ example in some detail. Afterwards we study D-branes in such models and also determine the zero-mode spectrum for some special values of the background gauge flux. In section 5.3 we apply asymmetric rotations to give an alternative derivation of the propagator on the disc with mixed Neumann-Dirichlet boundary conditions. In the final section of this chapter we apply all our techniques to the explicit construction of a $\mathbb{Z}_3 \times \widehat{\mathbb{Z}}_3$ orientifold containing D-branes with mixed boundary conditions.

Remark: If the string scale α' is not written explicitly, we have set it to one.

5.2 D-branes in asymmetric orbifolds

In this section we investigate in which way open strings manage to implement left-right asymmetric symmetries. Naively, one might think that asymmetric symmetries are an issue only in the closed string sector, as open strings can be obtained by projecting onto the left-right symmetric part of the space-time. However, historically just requiring the left-right asymmetric symmetry under T-duality on a circle led to the discovery of D-branes. This T-duality acts on the space-time coordinates as

$$(X_L, X_R) \to (-X_L, X_R) \tag{5.1}$$

Thus, the open string sector deals with T-duality by giving rise to a new kind of boundary condition leading in this case to the well known Dirichlet boundary condition. Compactifying on a higher dimensional torus T^d, in general with non-zero B-fields, the T-duality group gets enlarged, so that one may ask what the image of Neumann boundary conditions under these actions actually is.

In the course of this chapter we restrict ourselves to the two-dimensional torus T^2 and direct products thereof. For concreteness consider Type IIB compactified on a T^2 with complex coordinate $Z = X_1 + iX_2$ allowing a discrete $\mathbb{Z}_N$ symmetry acting as

$$\Theta : (Z_L, Z_R) \to \left(e^{i\theta} Z_L, e^{i\theta} Z_R\right) \tag{5.2}$$

with $\theta = 2\pi/N$. The essential observation is that performing a usual T-duality operation in the x_1-direction[4]

$$T : (Z_L, Z_R) \to (-\bar{Z}_L, Z_R) \tag{5.3}$$

yields an asymmetric action on the T-dual torus $\hat{T}^2$

$$\hat{\Theta} = T\Theta T^{-1} : (Z_L, Z_R) \to \left(e^{-i\theta} Z_L, e^{i\theta} Z_R\right) \tag{5.4}$$

The aim of this chapter is to investigate the properties of asymmetric orbifolds defined by actions like (5.4). The strategy we will follow is depicted in the following commuting diagram (figure 5.1): In order to obtain the features of

[4]This T-duality is the same as the D-duality introduced in section 2.2.2.1 (p. 38).

$$\begin{tikzcd}
\text{Type IIA} \arrow[r, "R\leftrightarrow\alpha'/R"] \arrow[d, "\hat\Theta"] & \text{Type IIB} \arrow[d, "\Theta"] \\
\text{Type IIA}/\mathbb{Z}_N \arrow[r, "R\leftrightarrow\alpha'/R"'] & \text{Type IIB}/\mathbb{Z}_N
\end{tikzcd}$$

Figure 5.1: T-duality relation

the asymmetric orbifold, concerning some questions it is appropriate to directly apply the asymmetric rotation $\hat\Theta$. For other questions it turns that it is better to first apply a T-duality and then perform the symmetric rotation Θ and translate the result back via a second T-duality.

5.2.1 Definition of the T-dual torus

The fundamentals underlying this section can be found in the review of Giveon, Porrati and Rabinovici [34]. The first step is to define the T-dual torus $\hat T^2$ allowing indeed an asymmetric action (5.4). Let the torus T^2 be defined by the following two vectors

$$e_1 = R_1, \qquad e_2 = R_2\, e^{i\alpha} \tag{5.5}$$

so that the complex and Kähler structures are given by

$$\begin{aligned} \tau &= \frac{e_2}{e_1} = \frac{R_2}{R_1}\, e^{i\alpha}, \\ \rho &= B_{12} + iR_1R_2\sin\alpha \end{aligned} \tag{5.6}$$

The left and right moving zero-modes, i.e. Kaluza-Klein and winding modes, can be written in the following form

$$\begin{aligned} p_L &= \frac{1}{i\sqrt{\tau_2\rho_2}}\left[\tau\, m_1 - m_2 - \bar\rho(n_1 + \tau\, n_2)\right], \\ p_R &= \frac{1}{i\sqrt{\tau_2\rho_2}}\left[\tau\, m_1 - m_2 - \rho(n_1 + \tau\, n_2)\right] \end{aligned} \tag{5.7}$$

Applying T-duality in the x_1-direction exchanges the complex-structure and the Kähler modulus yielding the torus $\hat T^2$ defined by the vectors[5]

$$\hat e_1 = \frac{1}{R_1}, \qquad \hat e_2 = \frac{B_{12}}{R_1} + iR_2\sin\alpha \tag{5.8}$$

and the two-form flux

$$\hat B_{12} = \frac{R_2}{R_1}\cos\alpha \tag{5.9}$$

For the Kaluza-Klein and winding modes we get

$$\begin{aligned} p_L &= -\frac{1}{i\sqrt{\hat\tau_2\hat\rho_2}}\left[\hat\tau\, n_1 + m_2 - \hat{\bar\rho}(m_1 + \hat\tau\, n_2)\right], \\ p_R &= -\frac{1}{i\sqrt{\hat\tau_2\hat\rho_2}}\left[\hat\tau\, n_1 + m_2 - \hat\rho(m_1 + \hat\tau\, n_2)\right] \end{aligned} \tag{5.10}$$

[5] A quantity describing the T-dual torus $\hat T^2$ is denoted by a hat ($\hat{\ }$) on it.

from which we deduce the relation of the Kaluza-Klein and winding quantum numbers

$$\widehat{m}_1 = -n_1, \quad \widehat{m}_2 = m_2, \quad \widehat{n}_1 = -m_1, \quad \widehat{n}_2 = n_2 \tag{5.11}$$

If the original lattice of T^2 allows a crystallographic action of a $\mathbb{Z}_N$ symmetry, then the T-dual Narain-lattice of $\hat{T}^2$ does allow a crystallographic action of the corresponding asymmetric $\widehat{\mathbb{Z}}_N$ symmetry. In view of the orientifold model studied in section 5.4, we present the $\mathbb{Z}_3$ case as an easy example.

5.2.2 The $\widehat{\mathbb{Z}}_3$ torus

In this section we shortly recall the definition of the $\mathbb{Z}_3$ and its T-dual $\widehat{\mathbb{Z}}_3$ that was given in section 2.2.2.1. One starts with the $\mathbb{Z}_3$ lattice defined by the basis vectors

$$e_1^{\mathbf{A}} = R, \qquad e_2^{\mathbf{A}} = R\left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right) \tag{5.12}$$

and arbitrary B-field. The complex-structure and Kähler moduli are

$$\begin{aligned} \tau^{\mathbf{A}} &= \frac{1}{2} + i\frac{\sqrt{3}}{2}, \\ \rho^{\mathbf{A}} &= B_{12} + iR^2\frac{\sqrt{3}}{2} \end{aligned} \tag{5.13}$$

This lattice has the additional property that it allows a crystallographic action of the reflection at (and consequently: along) the x_2-axis, $\bar{\sigma}$. This was important for the study of $\Omega\bar{\sigma}$ orientifolds in [56]. We call this lattice a "lattice of type **A**". Recall from [56], that under $\Omega\bar{\sigma}$ all three $\mathbb{Z}_3$ fixed points are left invariant. For zero B-field one obtains for instance for the T-dual **A** lattice

$$\hat{e}_1^{\mathbf{A}} = \frac{1}{R}, \qquad \hat{e}_2^{\mathbf{A}} = iR\frac{\sqrt{3}}{2} \tag{5.14}$$

and $\hat{b}^{\mathbf{A}} = 1/2$. That this rectangular lattice features an asymmetric $\widehat{\mathbb{Z}}_3$ symmetry and that all three "fixed points" of the $\widehat{\mathbb{Z}}_3$ are left invariant under Ω is not obvious at all. This shows already how T-duality can give rise to fairly non-trivial results.

As we have already shown in [56] (cf. section 2.2.2.2) there exists a second $\mathbb{Z}_3$ lattice, called type **B**, allowing a crystallographic action of the reflection $\bar{\sigma}$, too. The basis vectors are given by

$$e_1^{\mathbf{B}} = R, \qquad e_2^{\mathbf{B}} = \frac{R}{2} + i\frac{R}{2\sqrt{3}} \tag{5.15}$$

with arbitrary B-field leading to the complex-structure and Kähler moduli

$$\begin{aligned} \tau^{\mathbf{B}} &= \frac{1}{2} + i\frac{1}{2\sqrt{3}}, \\ \rho^{\mathbf{B}} &= B_{12} + i\frac{R^2}{2\sqrt{3}} \end{aligned} \tag{5.16}$$

For the $\mathbf{B}$ lattice only one $\mathbb{Z}_3$ fixed point is invariant under $\Omega\bar{\sigma}$, the remaining two are interchanged. For $B_{12} = 0$ the T-dual lattice is defined by

$$\hat{e}_1^{\mathbf{B}} = \frac{1}{R}, \qquad \hat{e}_2^{\mathbf{B}} = i\frac{R}{2\sqrt{3}} \tag{5.17}$$

with $\hat{b}^{\mathbf{B}} = 1/2$. It is a non-trivial consequence of T-duality that only one of the three $\widehat{\mathbb{Z}}_3$ "fixed points" is left invariant under Ω.

If one requires the lattices to allow simultaneously a symmetric $\mathbb{Z}_3$ and an asymmetric $\widehat{\mathbb{Z}}_3$ action one is stuck at the self-dual point $\tau = \rho$ yielding $R = 1$ and $B_{12} = 1/2$. Note, that this is precisely the root lattice of the $SU(3)$ Lie algebra. Since now we are equipped with lattices indeed allowing a crystallographic action of asymmetric $\widehat{\mathbb{Z}}_N$ operations, we can move forward to discuss their D-brane contents.

5.2.3 Asymmetric rotations of D-branes

In order to divide a string theory by some discrete group we first have to make sure that the theory is indeed invariant. For the open string sector this means that the D-branes also have to be arranged in such a way that they reflect the discrete symmetry. Thus, for instance we would like to know what the image of a D0-brane under an asymmetric rotation is. In the compact case we can ask this question for the discrete $\widehat{\mathbb{Z}}_N$ rotations defined in the last subsection, but we can also pose it quite generally in the non-compact case using a continuous asymmetric rotation

$$\begin{pmatrix} X'_{1,L} \\ X'_{2,L} \end{pmatrix} = \begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix} \begin{pmatrix} X_{1,L} \\ X_{2,L} \end{pmatrix} \qquad \begin{pmatrix} X'_{1,R} \\ X'_{2,R} \end{pmatrix} = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix} \begin{pmatrix} X_{1,R} \\ X_{2,R} \end{pmatrix} \tag{5.18}$$

As outlined already in the beginning of section 5.2 (see figure 5.1), instead of acting with the asymmetric rotation on the Dirichlet boundary conditions of the D0-brane, it is equivalent to go to the T-dual picture, apply first a symmetric rotation on the branes and then perform a T-duality transformation in the x_1-direction. In the T-dual picture the D0-brane becomes a D1-brane filling only the x_1-direction. Thus, the open strings are of Neumann type in the x_1-direction and of Dirichlet type in the x_2-direction. The asymmetric rotation becomes a symmetric rotation, which simply rotates the D1-brane by an angle ϕ in the x_1-x_2 plane. Thus, after the rotation the D1 boundary conditions in these two directions read

$$\begin{aligned} \partial_\sigma X_1 + \tan\phi\, \partial_\sigma X_2 = 0, \\ \partial_\tau X_2 - \tan\phi\, \partial_\tau X_1 = 0 \end{aligned} \tag{5.19}$$

If we are on the torus T^2 there is a distinction between values of ϕ, for which the rotated D1-brane intersects a lattice point, and values of ϕ, for which the D1-brane densely covers the entire T^2. In the first case, one still obtains quantized Kaluza-Klein and winding modes as computed in [108].

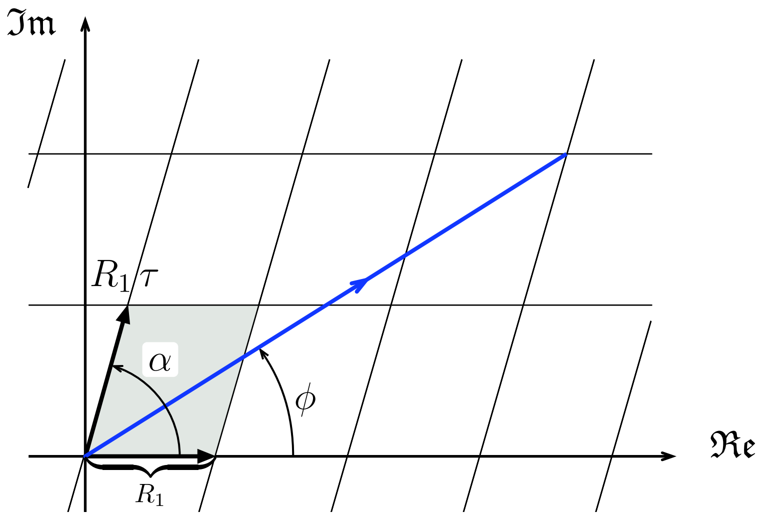

Figure 5.2: D1-brane (blue line) with wrapping numbers $(n, m) = (3, 2)$ on T^2. Note that the minimal distance of two such D-branes is smaller than the torus spacing R_1 and $R_2 \equiv R_1|\tau|$ while the length is larger.

If the D1-brane runs n-times around the e_1 circle and m times around the e_2 circle until it intersects a lattice point, the relation[6]

$$\cot\phi = \cot\alpha + \frac{n}{m\tau_2} \tag{5.20}$$

holds. As an example we show in figure 5.2 a rotated D1-brane with $n = 3$ and $m = 2$. In the following we will mostly consider D-branes of the first kind, which we will call rational D-branes. Finally, T-duality in the x_1-direction has the effect of exchanging $\partial_\sigma X_1 \leftrightarrow -\partial_\tau X_1$, leading to the boundary conditions [119]

$$\begin{aligned} \partial_\sigma X_1 + \cot\phi\, \partial_\tau X_2 &= 0, \\ \partial_\sigma X_2 - \cot\phi\, \partial_\tau X_1 &= 0 \end{aligned} \tag{5.21}$$

As emphasized already, one could also perform the asymmetric rotation directly on the Dirichlet boundary conditions for the D0-brane and derive the same result. Thus, we conclude that an asymmetric rotation turns a D0-brane into a D2-brane with mixed boundary conditions. The last statement is the main result of this chapter. As has been discussed intensively after the talks of Witten and Seiberg at the *Strings* conference in 1999 (and their related paper [83]), mixed boundary conditions arise from open strings traveling in a background with non-trivial two-form flux, B, or non-trivial gauge flux, F,

$$\begin{aligned} \partial_\sigma X_1 + (B+F)_{12}\partial_\tau X_2 &= 0, \\ \partial_\sigma X_2 - (B+F)_{12}\partial_\tau X_1 &= 0 \end{aligned} \tag{5.22}$$

[6] m and n are *co-prime*. If m and n would have a greatest common devisor p, this would mean that the brane is wrapped p times around the one-cycle defined by $(m/p, n/p)$.

Thus, we can generally identify

$$\cot\phi = \mathcal{F} = B + F \tag{5.23}$$

which in the rational case becomes (note that $\cot\phi$ is not necessarily rational)

$$\cot\phi = \cot\alpha + \frac{n}{m\tau_2} = B + F \tag{5.24}$$

Since the B field is related to the shape of the torus T^2 and the F field to the D-branes, from (5.24) we extract the following identifications[7]

$$B = \cot\alpha, \qquad F = \frac{n}{m\tau_2} \tag{5.25}$$

In section 5.3 we will further elaborate the relation between asymmetric rotations and D-branes with mixed boundary conditions. We will present an alternative derivation of some of the non-commutativity properties known for such boundary conditions. In the remainder of this section we will focus our attention on the momentum- and zero-mode spectrum for open strings stretched between D-branes with mixed boundary conditions. In particular, we will demonstrate that in the compact case open strings stretched between identical rational D-branes do have a non-trivial momentum-mode spectrum. This is in sharp contrast to some statements in the literature [120] saying that Neumann boundary conditions allow Kaluza-Klein momentum, Dirichlet boundary conditions allow non-trivial winding but general mixed D-branes do have neither of them.

5.2.4 Kaluza-Klein and winding modes, zero mode degeneracy

In this section we will calculate the spectrum of the linear modes for two dimensional D-branes with $U(1)$-F fluxes and for the T-dual configuration i.e. one dimensional D-branes on a T^2. The results are easily generalized to torus compactifications of the more general form:

$$\mathcal{M}_{10} = \mathbb{R}^{1,9-2d} \times \prod_{j=1}^{d} T^2_{(j)} \tag{5.26}$$

We will also explain the zero mode degeneracy that appears if the F fluxes are different at the two ends of a string. In the T-dual picture this degeneracy corresponds to the topological intersection number of two branes on a torus.

5.2.4.1 D2-branes with F-flux on T^2

Since we can not easily visualize a D-brane with mixed boundary conditions, we first determine the zero-mode spectrum in the closed string tree channel and then transform the result into the open string loop channel. In contrast to the underlying publication [1], we have now a direct method to quantize

[7] The F given in the following formula is written in the euclidian basis. In terms of a lattice basis for T^2 it is simply: $F = n/m$. m and n are co-prime.

open strings with mixed boundary conditions in toroidal compactifications (cf. chap. 4). However, we first calculate the spectrum by the use of boundary state formalism. After this, we use the results of the canonical quantization of the preceeding chapter, to derive an independent mass formula. These mass formulæ perfectly agree for the case $m = 1$ (cf. eq. (5.24)) without further explanation.

However for $m \neq 1$ (or fractional valued F-flux) there is a slight mismatch if we do not make further assumptions. In section 4.2.1.3 it was argued, that F/α' has to be integer-valued (in a basis of the torus lattice Γ^d). It was however mentioned that fractional valued F/α' corresponds to multiply wrapped branes, i.e. the lattice Γ^d of the closed string sector should be substituted by a bigger lattice $\tilde{\Gamma}^d$, s.th. F/α' is integer valued in a basis of $\tilde{\Gamma}^d$. Alternatively, we can demand the string wave function to be invariant only under shifts $\in m2\pi\Gamma^d$. By this we get perfect agreement even in the case $m \neq 1$.

We are looking for boundary states (see also [121]) in the closed string theory satisfying the following boundary state conditions

$$\begin{aligned} [\partial_\tau X_{1,cl} + \cot\phi\, \partial_\sigma X_{2,cl}]\, |B\rangle &= 0, \\ [\partial_\tau X_{2,cl} - \cot\phi\, \partial_\sigma X_{1,cl}]\, |B\rangle &= 0 \end{aligned} \tag{5.27}$$

Rewriting (5.27) in terms of the complex coordinate the boundary condition reads

$$[\partial_\tau Z_{\rm cl} - i\cot\phi\, \partial_\sigma Z_{\rm cl}]\, |B\rangle = 0 \tag{5.28}$$

There is an analogous condition for the hermitian conjugate field $\bar{Z}_{\rm cl}$. Using the mode expansion

$$\begin{aligned} Z_{\rm cl} &= \frac{z_0}{2} + \frac{1}{2}(p_L + p_R)\tau + \frac{1}{2}(p_L - p_R)\sigma \\ &\qquad + \frac{i}{\sqrt{2}} \sum_{n\neq 0} \left(\frac{\alpha_n}{n} e^{-in(\tau+\sigma)} + \frac{\tilde{\alpha}_n}{n} e^{-in(\tau-\sigma)} \right), \\ \bar{Z}_{\rm cl} &= \frac{\bar{z}_0}{2} + \frac{1}{2}(\bar{p}_L + \bar{p}_R)\tau + \frac{1}{2}(\bar{p}_L - \bar{p}_R)\sigma \\ &\qquad - \frac{i}{\sqrt{2}} \sum_{n\neq 0} \left(\frac{\bar{\alpha}_n}{n} e^{-in(\tau+\sigma)} + \frac{\bar{\tilde{\alpha}}_n}{n} e^{-in(\tau-\sigma)} \right) \end{aligned} \tag{5.29}$$

one obtains

$$\begin{aligned} [(p_L + p_R) - i\cot\phi\,(p_L - p_R)]\, |B\rangle &= 0, \\ \left[\alpha_n + e^{2i\phi}\tilde{\alpha}_{-n}\right] |B\rangle &= 0 \end{aligned} \tag{5.30}$$

with similar conditions for the fermionic modes. Inserting (5.10) and (5.11) into the first equation of (5.30) one can solve for the Kaluza-Klein and winding modes

$$\widehat{m}_1 = -\frac{n}{m}\widehat{n}_2, \qquad \widehat{m}_2 = \frac{n}{m}\widehat{n}_1 \tag{5.31}$$

giving rise to the following momentum mode spectrum:

$$M^2_{\rm cl}(\vec{r}) = \frac{|r_1 + r_2\,\hat{\tau}|^2}{\hat{\tau}_2} \frac{|n + m\,\hat{\rho}|^2}{\hat{\rho}_2}, \qquad \vec{r} \in \mathbb{Z}^2 \tag{5.32}$$

with $r, s \in \mathbb{Z}$. We observe that this agrees with the spectrum derived in [108] by employing T-duality. The oscillator part of a bosonic boundary state satisfying (5.30) is given by

$$|B\rangle_{(n,m)} = \sum_{r,s\in\mathbb{Z}} \exp\Big(\sum_{n\in\mathbb{Z}} \frac{1}{n} e^{2i\phi} \alpha_{-n}\tilde{\alpha}_{-n}\Big)|r,s\rangle_{(n,m)} \tag{5.33}$$

Using this boundary state we compute the tree channel annulus partition function. Transforming the result via a modular transformation into loop channel, we can extract the momentum-mode contribution and conclude that open strings stretching between identical rational D-branes carry non-vanishing linear modes giving rise to masses

$$M^2_{\text{op}}(\vec{s}) = \frac{|s_1 + s_2\,\hat{\tau}|^2}{\hat{\tau}_2} \frac{\hat{\rho}_2}{|n + m\,\hat{\rho}|^2}, \qquad \vec{s} \in \mathbb{Z}^2 \tag{5.34}$$

As announced, due to the results of the preceding chapter, we have an alternative way at hand to derive this formula. The mass formula for an open string on a lattice Γ^d (without Dirichlet bdy.-conditions) is given by formula (4.91) (p. 103). However we interpret m as the wrapping number of the D2 brane and require invariance only under $m2\pi\Gamma^2$ which is only a subgroup of the closed string torus $2\pi\Gamma^2$. This means that we consider the m^2-fold cover of the torus defined by $2\pi\Gamma^2$. The single valuedness condition of the wavefunction under $m2\pi\Gamma^2$ now reads (cf. eq. (4.90), p. 103):

$$\sum_{i=1}^{2} e_i p^i(\vec{s}) = \Big(G \mp \mathcal{F}_{\frac{1}{2}}\Big)^{-1} \frac{1}{m} \sum_{i=1}^{2} s_i e^i \qquad \vec{s} \in \mathbb{Z}^2 \tag{5.35}$$

This leads via (4.44) (p. 94) to the mass-formula for open strings in F-field backgrounds:

$$M^2_{\text{op}}(\vec{s}) = \frac{\vec{s}^T\, G\vec{s}}{\det(mG) + (n + mB_{12})^2} \tag{5.36}$$

Using the definitions (2.56) and (2.57) (p. 36), we note that (5.34) and (5.36) agree.

Summarizing, we now have the means to compute annulus amplitudes for open strings stretched between different kinds of D-branes with rational mixed boundary conditions.

There is still the question of the number of *Landau levels* that occur if both field strength F_1 and F_2 do not agree. By the means of section (4.2.1.3) we have a concrete formula to calculate this degeneracy. However, we have to take into account, that the two branes appearing in $n_{i,j}$ have in general different wrapping numbers m_i and m_j. Therefore we expect that wavefunctions can only be invariant under translations $m_i m_j 2\pi\Gamma^2$ instead of being invariant under $2\pi\Gamma^2$. Taking this into account in the derivation of (4.94) (p. 103) we see that the degeneracy on each T^2 is given by:

$$n_{i,j} = (-1)\,\text{Pf}\,\big(m_i m_j (F_i + F_j)\big) = (n_j m_i - n_i m_j) \tag{5.37}$$

The Landau degeneracy (5.37) equals the *topological intersection* number in the T-dual picture where the two dimensional F-flux branes correspond to one-dimensional branes with non-trivial intersection.

We observed in chapter 2 that the NSNS B field has to be quantized, if we want to gauge the world sheet parity Ω (cf. eq. (2.40), p. 34). For the two-torus this means that $B_{12}/\alpha' = 1/2$. Ω exchanges the $\sigma = 0$ with the $\sigma = \pi$ boundary of the string. Thereby the τ-derivative changes sign. This can be absorbed in redefining $F_i \to -F_i$. (We have to include boundary conditions with reversed F-field.) For Ω to be a symmetry, the open string mass spectrum has to be invariant. From eq. (5.36) we deduce that the action of Ω for backgrounds with $B_{12}/\alpha' = 1/2$ is:

$$\Omega_{B_{12}=1/2}: \quad \begin{array}{rcl} n & \to & n' = n+m \\ m & \to & m' = -m \\ F = \frac{1}{m}\begin{pmatrix} 0 & n \\ -n & 0 \end{pmatrix} & \to & F' = \frac{1}{m}\begin{pmatrix} 0 & -(n+m) \\ n+m & 0 \end{pmatrix} \end{array} \tag{5.38}$$

For vanishing B-field we simply get:

$$\Omega_{B_{12}=0}: \quad \begin{array}{rcl} n & \to & n' = n \\ m & \to & m' = -m \end{array} \tag{5.39}$$

This has an important consequence (especially for model-building, as we will see in the next chapter): the difference between the multiplicity of states

$$n_{ij} - n_{i\Omega(j)} \in 2\mathbb{Z} \qquad \text{for } B_{12} = 0 \tag{5.40}$$

$$n_{ij} - n_{i\Omega(j)} \in \mathbb{Z} \qquad \text{for } B_{12} = \alpha'/2 \tag{5.41}$$

has always to be an even number for vanishing B field, while it can be both odd and even for $B_{12} = \alpha'/2$.

As an example, we discuss the $\widehat{\mathbb{Z}}_3$ case in some more detail.

5.2.4.1.1 D-branes in the asymmetric $\widehat{\mathbb{Z}}_3$ orbifold Consider the $\widehat{\mathbb{Z}}_3$ lattice of type **A** and start with a D_1-brane with pure Dirichlet boundary conditions ($\phi = 0$)

$$\begin{aligned} \partial_\tau X_1 &= 0, \\ \partial_\tau X_2 &= 0 \end{aligned} \tag{5.42}$$

Successively applying the asymmetric $\widehat{\mathbb{Z}}_3$ this D-brane is mapped to a mixed D_2-brane with boundary conditions ($\phi = 2\pi/3$)

$$\begin{aligned} \partial_\sigma X_1 - \frac{1}{\sqrt{3}}\partial_\tau X_2 &= 0, \\ \partial_\sigma X_2 + \frac{1}{\sqrt{3}}\partial_\tau X_1 &= 0 \end{aligned} \tag{5.43}$$

and a mixed D_3-brane with boundary conditions ($\phi = -2\pi/3$)

$$\begin{aligned} \partial_\sigma X_1 + \frac{1}{\sqrt{3}}\partial_\tau X_2 &= 0, \\ \partial_\sigma X_2 - \frac{1}{\sqrt{3}}\partial_\tau X_1 &= 0 \end{aligned} \tag{5.44}$$

In the orbifold theory these three kinds of D-branes are identified. This reflects that their background fields are being identified according to

$$\mathcal{F} \sim \mathcal{F} + \frac{1}{\sqrt{3}} \tag{5.45}$$

or equivalently

$$\phi \sim \phi + \frac{2\pi}{3} \tag{5.46}$$

The two coordinates X_1 and X_2 yield the following contribution to the annulus partition function for open strings stretched between identical D-branes

$$A_{ii}^{\alpha\beta} = \frac{\vartheta\begin{bmatrix}\alpha\\ \beta\end{bmatrix}}{\eta^3}\left(\sum_{r\in\mathbb{Z}} e^{-2\pi t\frac{r^2}{R^2}}\right)\left(\sum_{s\in\mathbb{Z}} e^{-2\pi t\frac{3s^2}{4R^2}}\right) \tag{5.47}$$

independent of $i \in \{1, 2, 3\}$. Open strings stretched between different kinds of D-branes give rise to shifted moding and yield the partition function

$$A_{i,i+1}^{\alpha\beta} = n_{i,i+1}\frac{\vartheta\begin{bmatrix}\frac{1}{3}+\alpha\\ \beta\end{bmatrix}}{\vartheta\begin{bmatrix}\frac{1}{3}+\alpha\\ \frac{1}{2}\end{bmatrix}} \tag{5.48}$$

which looks like a twisted open string sector. As we know from [56, 117, 57] (and the preceding discussion) we have to take into account extra multiplicities, $n_{i,i+1}$, which have a natural geometric interpretation as multiple intersection points of D-branes at angles in the T-dual picture. By this reasoning we find that for the **A** type lattice the extra factor is one. However, for the three D-branes generated by $\widehat{\mathbb{Z}}_3$ when one starts with a D-brane with pure Neumann boundary conditions, $\phi \in \{\pi/2, \pi/6, -\pi/6\}$, T-duality tells us that there must appear an extra factor of three in front of the corresponding annulus amplitude (5.48). In the orientifold construction presented in section 5.4 these multiplicities are important to give consistent models.

5.2.4.2 D1-branes on T^2

In this section we will calculate the masses of the linear (i.e. momentum) modes of a D-brane that has Dirichlet bdy.-cond. in one direction and Neumann conditions in the perpendicular direction. This situation is depicted in figure 5.2 (p. 123). We denote the vector tangential to the brane by $\vec{t}$ and the one normal to the brane by $\vec{n}$:

$$\vec{t} \equiv ne_1 + me_2 \qquad \vec{n} \equiv (me_1^* - ne_2^*)\left(\left(\begin{smallmatrix}m\\ -n\end{smallmatrix}\right)^T G^*\left(\begin{smallmatrix}m\\ -n\end{smallmatrix}\right)\right)^{-1/2} \tag{5.49}$$

With these conventions, we see that the linear part of the field X becomes (cf. (4.48), p. 95):

$$X_{\text{lin}} = \tau\,\vec{t}\,p_{\vec{t}} + \sigma\,\vec{n}\,p_{\vec{n}} \tag{5.50}$$

$\vec{n}p_{\vec{n}}$ points perpendicular to the D-brane. Therfore $\pi p_{\vec{n}}$ must be an integer multiple of the minimal distance of two branes on the torus. Therfore we get:[8]

$$p_{\vec{n}}(s_2) = s_2 \left(\left(\begin{smallmatrix} m \\ -n \end{smallmatrix} \right)^T G^* \left(\begin{smallmatrix} m \\ -n \end{smallmatrix} \right) \right)^{-1/2} \qquad s_2 \in \mathbb{Z} \tag{5.51}$$

The momentum in the τ-direction is conserved. It reads (c.f. eq. (4.79), p. 101):

$$\Pi_{\vec{t}} = \frac{1}{2\pi} \left(\|\vec{t}\| p_{\vec{t}} + \overbrace{\frac{\vec{t}}{\|\vec{t}\|} B_{12} \vec{n}}^{=B_{12}} p_{\vec{n}}(s_2) \right) \tag{5.52}$$

Requiring invariance of the wavefunction under a translation by $2\pi\|\vec{t}\|$ in the $\vec{t}$ direction leads to

$$p_{\vec{t}}(s_1, s_2) = \left(\left(\begin{smallmatrix} n \\ m \end{smallmatrix} \right)^T G \left(\begin{smallmatrix} n \\ m \end{smallmatrix} \right) \right)^{-1/2} \left(s_1 - B_{12}\, p_{\vec{n}}(s_2) \right) \qquad s_1, s_2 \in \mathbb{Z} \tag{5.53}$$

The resulting mass^2 formula takes the form:

$$M^2_{\mathrm{op}}(\vec{s}) = \frac{\vec{s}^T \left(\begin{smallmatrix} 1 & -B_{12} \\ -B_{12} & \det(G)+B_{12}^2 \end{smallmatrix} \right) \vec{s}}{\left(\begin{smallmatrix} n \\ m \end{smallmatrix} \right) G \left(\begin{smallmatrix} n \\ m \end{smallmatrix} \right)}, \qquad \vec{s} \in \mathbb{Z}^2 \tag{5.54}$$

This formula is T-dual (under the duality denoted by "D" in section 2.2.2.1, eq. (2.58)) to the mass formula (5.36) and (5.34). Of course we can rewrite (5.54) in terms of the complex-structure τ and the Kähler structure ρ:

$$M^2_{\mathrm{op}}(\vec{s}) = \frac{|s_1 + s_2\, \rho|^2}{\rho_2} \, \frac{\tau_2}{|n + m\, \tau|^2}, \qquad \vec{s} \in \mathbb{Z}^2 \tag{5.55}$$

5.3 Asymmetric rotations and non-commutative geometry

In section 5.2 we have pointed out that on T^2 or $\mathbb{R}^2$ D-branes with mixed boundary conditions can be generated by simply applying an asymmetric rotation to an ordinary D-brane with pure Neumann or Dirichlet boundary conditions. Thus, it should be possible to rederive earlier results for the two-point function on the disc

$$\langle X_i(z)\, X_j(z') \rangle \tag{5.56}$$

for the operator product expansion (OPE) between vertex operators on the boundary

$$e^{ipX}(\tau)\, e^{iqX}(\tau') \tag{5.57}$$

by applying an asymmetric rotation on the corresponding quantities for open strings ending on D0-branes in flat space-time.

---

[8] To calculate the minimal distance between two D1-branes we use an elementary theorem of number theory: *The units in the ring* $\mathbb{Z}/n\mathbb{Z}$ *consist of those residue classes* mod $n\mathbb{Z}$ *which are represented by integers* $m \neq 0$ *and prime to* n (cf. [96], chapter II, §2). This implies that given two integers n and m which are prime w.r.t. each other there exist two integers a, b s.th.: $an + bm = 1$.

5.3.1 Two-point function on the disc

The two-point function on the disc for both X_1 and X_2 of Dirichlet type reads

$$\begin{aligned}\langle X_i(z)\, X_j(z')\rangle &= -\alpha'\delta_{ij}\left(\ln|z-z'| - \ln|z-\bar{z}'|\right)\\ &= -\alpha'\delta_{ij}\frac{1}{2}\left(\ln(z-z') + \ln(\bar{z}-\bar{z}') - \ln(z-\bar{z}') - \ln(\bar{z}-z')\right)\end{aligned} \tag{5.58}$$

from which, formally using

$$X_i(z) = X_{i,L}(z) + X_{i,R}(\bar{z}) \tag{5.59}$$

we can directly read off the individual contributions from the left- and right-movers. Performing the asymmetric rotation

$$X_L \to A\, X_L, \qquad X_R \to A^T X_R \tag{5.60}$$

where A denotes an element of $SO(2)$, leads to the following expression for the propagator in the rotated coordinates

$$\begin{aligned}\langle X_i(z)\, X_j(z')\rangle = &-\alpha'\delta_{ij}\ln|z-z'| - \alpha'\delta_{ij}\left(\sin^2\phi - \cos^2\phi\right)\ln|z-\bar{z}'|\\ &-\alpha'\epsilon_{ij}\sin\phi\,\cos\phi\,\ln\left(\frac{z-\bar{z}'}{\bar{z}-z'}\right)\end{aligned} \tag{5.61}$$

This expression agrees precisely with the propagator derived in [105] with the identification

$$\mathcal{F} = \begin{pmatrix} 0 & \cot\phi \\ -\cot\phi & 0 \end{pmatrix} \tag{5.62}$$

Thus, by applying an asymmetric rotation we have found an elegant and short way of deriving this propagator without explicit reference to the boundary conditions or the background fields. Moreover, since the commutative D0-brane is related in this smooth way to a non-commutative D2-brane, it is suggesting that also both effective theories arising on such branes are related by some smooth transformation. Such an explicit map between the commuting and the non-commuting effective gauge theories has been determined in [83].

5.3.2 The OPE of vertex operators

In this subsection we apply an asymmetric rotation also to the operator product expansion of tachyon vertex operators $\mathcal{O}(z) = e^{ipX}(z)$ on the boundary. Of course this OPE is a direct consequence of the correlator (5.61) restricted to the boundary, but nevertheless we would like to see whether we can generate the non-commutative $*$-product directly via an asymmetric rotation. Taking care of the left- and right-moving contributions in the OPE between vertex operators living on a pure Dirichlet boundary we can write for $|z| > |z'|$

$$e^{ipX}(z)\, e^{iqX}(z') = \frac{(z-z')^{\frac{\alpha'}{2}p_L q_L}\,(\bar{z}-\bar{z}')^{\frac{\alpha'}{2}p_R q_R}}{(z-\bar{z}')^{\frac{\alpha'}{2}p_L q_R}\,(\bar{z}-z')^{\frac{\alpha'}{2}p_R q_L}}\; e^{i(p+q)X}(z') + \ldots \tag{5.63}$$

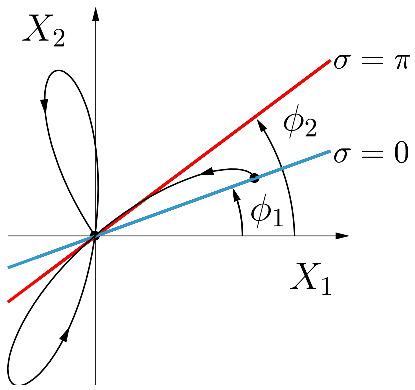

Figure 5.3: String attached to branes at angles. The relative angle is $\phi_2 - \phi_1$. The boundary conditions are given by eq. (5.67) and (5.68) (Cf. fig. 1.6, p. 22).

Now we apply an asymmetric rotation (5.60) together with

$$\begin{aligned} p_L &\to A\, p_L, \qquad & p_R &\to A^T p_R, \\ q_L &\to A\, q_L, \qquad & q_R &\to A^T q_R \end{aligned} \tag{5.64}$$

and, after all, identifying $p_L = p_R$, $q_L = q_R$ we obtain

$$e^{ipX}(z)\, e^{iqX}(z') = \frac{\left[(z-z')(\bar z - \bar z')\right]^{\frac{\alpha'}{2}pq}}{\left[(z-\bar z') \times (\bar z - z')\right]^{\frac{\alpha'}{2}\cos(2\phi)\,pq}} \Big(\frac{z-\bar z'}{\bar z - z'}\Big)^{-\frac{\alpha'}{2}\epsilon_{ij}p_i q_j \sin(2\phi)}\; e^{i(p+q)X}(z') + \dots \tag{5.65}$$

Restricting (5.65) to the boundary and choosing the same branch cut as in [83] we finally arrive at

$$e^{ipX}(\tau)\, e^{iqX}(\tau') = (\tau-\tau')^{\alpha' pq(1+\sin^2\phi - \cos^2\phi)} \exp\big(-i\pi\alpha' \sin\phi\cos\phi\epsilon_{ij}p_i q_j\big)\; e^{i(p+q)X}(\tau') + \dots$$

This is precisely the OPE derived in [85,83]. It shows that it is indeed possible to derive the $*$-product $e^{ipX}(\tau)e^{iqX}(\tau') \sim e^{ipX} * e^{iqX}(\tau')$ directly via an asymmetric rotation, where the non-commutative algebra $\mathcal{A}$ of functions f and g is defined as

$$f * g = fg - i\pi\alpha' \sin\phi\, \cos\phi\, \epsilon_{ij}\, \partial_i f \partial_j g + \dots \tag{5.66}$$

5.3.3 The commutator of the coordinates

While the two-point function derived above already implies that the commutator of the coordinate fields is non-vanishing, i.e. the geometry on the D-brane non-commutative, we would like to rederive this result directly via studying D-branes with mixed boundary conditions, as well. This is done by the quantization of the bosonic coordinate fields of the open string. We start with the T-dual situation with two D-branes intersecting at an arbitrary angle $\phi_2 - \phi_1$ (see figure 5.3). The open string boundary conditions at $\sigma = 0$ are

$$\begin{aligned} \partial_\sigma X_1 + \tan\phi_1\, \partial_\sigma X_2 = 0, \\ \partial_\tau X_2 - \tan\phi_1\, \partial_\tau X_1 = 0 \end{aligned} \tag{5.67}$$

and at $\sigma = \pi$ we require

$$\begin{aligned} \partial_\sigma X_1 + \tan\phi_2\, \partial_\sigma X_2, = 0, \\ \partial_\tau X_2 - \tan\phi_2\, \partial_\tau X_1 = 0 \end{aligned} \tag{5.68}$$

The mode expansion satisfying these two boundary conditions looks like

$$\begin{aligned} X_1 = x_1 + & i\sqrt{\alpha'} \sum_{n\in\mathbb{Z}} \frac{\alpha_{n+\nu}}{n+\nu} e^{-i(n+\nu)\tau} \cos[(n+\nu)\sigma + \phi_1] + \\ & i\sqrt{\alpha'} \sum_{m\in\mathbb{Z}} \frac{\alpha_{m-\nu}}{m-\nu} e^{-i(m-\nu)\tau} \cos[(m-\nu)\sigma - \phi_1], \\ X_2 = x_2 + & i\sqrt{\alpha'} \sum_{n\in\mathbb{Z}} \frac{\alpha_{n+\nu}}{n+\nu} e^{-i(n+\nu)\tau} \sin[(n+\nu)\sigma + \phi_1] - \\ & i\sqrt{\alpha'} \sum_{m\in\mathbb{Z}} \frac{\alpha_{m-\nu}}{m-\nu} e^{-i(m-\nu)\tau} \sin[(m-\nu)\sigma - \phi_1] \end{aligned} \tag{5.69}$$

with $\nu = (\phi_2 - \phi_1)/\pi$. Using the usual commutation relation

$$[\alpha_{n+\nu}, \alpha_{m-\nu}] = (n+\nu)\, \delta_{m+n,0} \tag{5.70}$$

and the vanishing of the commutator of the center of mass coordinates x_1 and x_2 one can easily show that for D-branes at angles the general equal time commutator vanishes

$$[X_i(\tau,\sigma), X_j(\tau,\sigma')] = 0 \tag{5.71}$$

Therefore, the geometry of D-branes at angles, but without background gauge fields, is always commutative.

Performing T-duality in the x_1 direction one gets the two mixed boundary conditions for the open strings

$$\begin{aligned} \partial_\sigma X_1 + \cot\phi_1\, \partial_\tau X_2 = 0, \\ \partial_\sigma X_2 - \cot\phi_1\, \partial_\tau X_1 = 0 \end{aligned} \tag{5.72}$$

at $\sigma = 0$ and

$$\begin{aligned} \partial_\sigma X_1 + \cot\phi_2\, \partial_\tau X_2 = 0, \\ \partial_\sigma X_2 - \cot\phi_2\, \partial_\tau X_1 = 0 \end{aligned} \tag{5.73}$$

at $\sigma = \pi$. This is a very special case of what we considered in chapter 4. It corresponds to the following field strengths:

$$\mathcal{F}_1 = \begin{pmatrix} 0 & -\cot\phi_1 \\ \cot\phi_1 & 0 \end{pmatrix} \qquad \mathcal{F}_2 = \begin{pmatrix} 0 & \cot\phi_2 \\ -\cot\phi_2 & 0 \end{pmatrix} \tag{5.74}$$

In section 4.3 we proved that the commutator of the coordinate fields X takes the following values at its boundaries:

$$[X(\tau,\sigma), X(\tau,\sigma)]\,\big|_{\sigma\in\partial\mathcal{M}_j} = i2\pi\alpha' \frac{\mathcal{F}_j}{G - \mathcal{F}_j^2} \qquad j = 1,2 \tag{5.75}$$

This result was obtained for the two-torus [112, 84] as well.

At the end of this section let us briefly comment on the algebraic structure of the non-commutative torus we have obtained by the asymmetric rotation on the D-branes. As shown in the previous section, the tachyon vertex operator $\mathcal{O} = e^{ipX}(\tau)$ leads to a non-commutative algebra $\mathcal{A}$, defined in eq. (5.66). As explained in [83], the algebra $\mathcal{A}$ of tachyon vertex operators can be taken at either end of the open string. Therefore the open string states form a bi-module $\mathcal{A}\times\mathcal{A}'$, where $\mathcal{A}$ is acting on the boundary $\sigma = 0$ and $\mathcal{A}'$ on the boundary $\sigma = \pi$ of the open string. Specifically, for an open string whose first boundary $\sigma = 0$ is related to a D-brane with parameter ϕ_1 and whose second boundary $\sigma = \pi$ is attached to a D-brane with parameter ϕ_2, the algebra $\mathcal{A}$ of functions on the non-commutative torus is generated by

$$\begin{aligned} U_1 &= \exp\Big(iy_1 - \frac{2\pi^2\alpha'\mathcal{F}_1}{1+\mathcal{F}_1^2}(\partial/\partial y_2)\Big), \\ U_2 &= \exp\Big(iy_2 + \frac{2\pi^2\alpha'\mathcal{F}_1}{1+\mathcal{F}_1^2}(\partial/\partial y_1)\Big) \end{aligned} \tag{5.76}$$

which obey

$$U_1U_2 = \exp\Big(-2\pi i\frac{2\pi\alpha'\mathcal{F}_1}{1+\mathcal{F}_1^2}\Big)U_2U_1 \tag{5.77}$$

On the other hand, the algebra $\mathcal{A}'$ is generated by

$$\begin{aligned} \tilde{U}_1 &= \exp\Big(iy_1 + \frac{2\pi^2\alpha'\mathcal{F}_2}{1+\mathcal{F}_2^2}(\partial/\partial y_2)\Big), \\ \tilde{U}_2 &= \exp\Big(iy_2 - \frac{2\pi^2\alpha'\mathcal{F}_2}{1+\mathcal{F}_2^2}(\partial/\partial y_1)\Big) \end{aligned} \tag{5.78}$$

obeying

$$\tilde{U}_1\tilde{U}_2 = \exp\Big(2\pi i\frac{2\pi\alpha'\mathcal{F}_2}{1+\mathcal{F}_2^2}\Big)\tilde{U}_2\tilde{U}_1 \tag{5.79}$$

5.4 Asymmetric orientifolds

Another motivation for studying such asymmetric orbifolds arises in the construction of Type I vacua. In [56,57] we have considered so-called supersymmetric orientifolds with D-branes at angles in six and four space-time dimensions which in the six-dimensional case were defined as

$$\frac{\text{Type IIB on } T^4}{\{\Omega\bar{\sigma}, \Theta\}} \tag{5.80}$$

with $\bar{\sigma} : z_i \to -\bar{z}_i$, the z_i being the complex coordinates of the T^4. Upon T-dualities in the directions of their real parts one obtains an ordinary orientifold where, however, the space-time symmetry becomes asymmetric

$$\frac{\text{Type IIB on } \hat{T}^4}{\{\Omega, \hat{\Theta}\}} \tag{5.81}$$

In the entire derivation in section 5.2 we have identified the two constructions explicitly via T-duality, relating branes with background fields to branes at angles. While in the $\Omega\bar{\sigma}$ orientifolds Θ identified branes at different locations on the tori, $\hat{\Theta}$ now maps branes with different values of their background gauge flux upon each other. As the background fields determine the parameter which rules the non-commutative geometry, branes with different geometries are identified according to (5.45). However the non-commutativity is solely restricted to the compactification space, i.e.

$$\Big(\prod_i \hat{T}^2_{(i)}\Big)\Big/\widehat{\mathbb{Z}}_N \tag{5.82}$$

Each $\hat{T}^2_{(i)}$ has to admit the respective $\widehat{\mathbb{Z}}_N$ symmetry. The $\widehat{\mathbb{Z}}_3$ tori (2.74), p. 39 (**A**-torus) and (2.87), p. 41 (**B**-torus) both have a ρ_2 component of order one. As ρ_2 is the volume of the two-torus measured in terms of α', the compactification size is roughly speaking the string scale. No large volume of the asymmetric $\widehat{\mathbb{Z}}_3$ (and more general: $\widehat{\mathbb{Z}}_N$) orbifold exists. Even though we will get $SO(N)$ gauge groups in the $\mathbb{Z}_3^L \times \mathbb{Z}_3^R$ orientifold on T^4 this is not in contradiction with the observation that no non-commutative gauge theories with gauge-groups other than $U(n)$ seem to exist (c.f. [83, 122, 123]): The $\mathbb{Z}_3^L \times \mathbb{Z}_3^R$ orientifold does not admit a decompactification limit.

From the above mentioned identification it is now clear that the $\mathcal{N} = (0,1)$ supersymmetric asymmetric $\widehat{\mathbb{Z}}_N$ orientifolds (5.81) have the same one loop partition functions as the corresponding symmetric $\mathbb{Z}_N$ orientifolds (5.80). The only difference is that instead of D7-branes at angles, we introduce D9-branes with appropriate background fields. Thus, a whole class of asymmetric orientifolds has already been studied in the T-dual picture involving D-branes at angles. One could repeat the whole computation for the asymmetric orientifolds (5.81), getting of course identical results. Note, the model (5.81) is really a Type I vacuum, as Ω itself is gauged. Thus, in principle there exist the possibility that heterotic dual models exist. Of course, in six dimensions most models have more than one tensor-multiplet so that no perturbative heterotic dual model can exist. It would be interesting to look for heterotic duals for the four dimensional models discussed in [57].

5.4.1 Orientifolds on the $\big(T^2 \times T^2\big)\big/\big(\mathbb{Z}_3^L \times \mathbb{Z}_3^R\big)$ orbifold background

In the following we will construct the even more general six-dimensional $\mathbb{Z}_3 \times \widehat{\mathbb{Z}}_3$ orientifold

$$\frac{\text{Type IIB on } \hat{T}^4}{\{\Omega, \Theta, \hat{\Theta}\}}$$

which is T-dual to

$$\frac{\text{Type IIB on } T^4}{\{\Omega\bar{\sigma}, \hat{\Theta}, \Theta\}}$$

where in fact, as shown in section 5.2.2, the two tori are identical $T^4 = \hat{T}^4 = SU(3)^2$. The freedom to choose their complex-structures gives rise to a variety of three distinct models, which are denoted by **AA**, **AB**, **BB** as in [56]. Note, that the same orbifold group is generated by a pure left-moving $\mathbb{Z}_{3L}$, $\Theta_L = \hat{\Theta}\Theta^{-1}$, and a pure right-moving $\mathbb{Z}_{3R}$, $\Theta_R = \hat{\Theta}\Theta$. As was also shown in [53] this model actually has $\mathcal{N} = (1,1)$ supersymmetry, but one can get $\mathcal{N} = (0,1)$ supersymmetry by turning on non-trivial discrete torsion.

5.4.1.1 Tadpole cancellation

The computation of the various one-loop amplitudes is straightforward. For the loop channel Klein bottle amplitude we obtain

$$\begin{aligned} K^{(ab)} = \frac{16c}{12} \int_0^\infty \frac{dt}{t^4} \frac{1}{\eta^4} \Big[& \rho_{00}\, \Lambda^a \Lambda^b + \rho_{01} + \rho_{02} \\ & + n^{(ab)}_{\hat{\Theta},\Omega}\, \rho_{10} + n^{(ab)}_{\hat{\Theta},\Omega\Theta^2}\, \epsilon \rho_{11} + n^{(ab)}_{\hat{\Theta},\Omega\Theta}\, \bar{\epsilon}\rho_{12} \\ & + n^{(ab)}_{\hat{\Theta}^2,\Omega}\, \rho_{20} + n^{(ab)}_{\hat{\Theta}^2,\Omega\Theta^2}\, \bar{\epsilon}\rho_{21} + n^{(ab)}_{\hat{\Theta}^2,\Omega\Theta}\, \epsilon\rho_{22} \Big] \end{aligned} \tag{5.83}$$

where $c \equiv \mathrm{V}_6 / \left(8\pi^2\alpha'\right)^3$ and ϵ is a phase factor defining the discrete torsion. Furthermore we use functions ρ_{gh} that are adopted from [53]. We already expressed the torus partition function of the orbifold theory in terms of these functions (section 2.4). They are given in appendix A.3 (p. 208) together with their modular properties. g and h denote the twists resp. the projections in the partition function ρ_{gh}: $g, h \in \{(0, 1/3, -1/3), (0, 2/3, -2/3)\}$ for which we use the shorter notation $g, h \in \{0, 1, 2\}$. The index (ab) denotes the three possible choices of lattices, **AA**, **AB** and **BB**, and Λ^a are the zero mode contributions (5.34) to the partition function

$$\begin{aligned} \Lambda^{\mathbf{A}} &= \frac{1}{\eta^2} \sum_{m_1,m_2} e^{-\pi t \left[m_1^2 + \frac{4}{3}\left(\frac{m_1}{2} - m_2\right)^2 \right]} = \sum_{i=0}^{2} \chi_i \\ \Lambda^{\mathbf{B}} &= \frac{1}{\eta^2} \sum_{m_1,m_2} e^{-\pi t \left[m_1^2 + 12\left(\frac{m_1}{2} - m_2\right)^2 \right]} = \chi_0 \end{aligned} \tag{5.84}$$

χ_i are $SU(3)$ characters with argument $q = \exp(-4\pi t)$ (cf. eq. (2.113), p. 48). Finally, $n^{(ab)}_{\Sigma_1,\Sigma_2}$ denotes the trace of the action of Σ_2 on the fixed points in the Σ_1 twisted sector. Taking into account that the origin is the only common fixed point of $\mathbb{Z}_3$ and $\widehat{\mathbb{Z}}_3$, they can be determined to be

$$n^{(ab)}_{\hat{\Theta},\Omega} = \begin{cases} 9 & \text{for } (\mathbf{AA}) \\ 3 & \text{for } (\mathbf{AB}) \\ 1 & \text{for } (\mathbf{BB}) \end{cases} \tag{5.85}$$

and

$$n^{(ab)}_{\hat{\Theta},\Omega\Theta} = n^{(ab)}_{\hat{\Theta}^2,\Omega\Theta^2} = \begin{cases} -3 & \text{for } (\mathbf{AA}) \\ i\sqrt{3} & \text{for } (\mathbf{AB}) \\ 1 & \text{for } (\mathbf{BB}) \end{cases} \tag{5.86}$$

The remaining numbers are given by complex conjugation of (5.86). Applying a modular transformation to (5.83) yields the tree channel Klein bottle amplitude

$$\begin{aligned}\widetilde{K}^{(ab)} = 2\frac{32c}{3}\int_0^\infty dl \frac{1}{\eta^4}\Big[&\rho_{00}\,\tilde{\Lambda}^a\tilde{\Lambda}^b + \frac{1}{3}n^{(ab)}_{\hat{\Theta}^2,\Omega}\,\rho_{01} + \frac{1}{3}n^{(ab)}_{\hat{\Theta},\Omega}\,\rho_{02} \\ &+ 3\rho_{10} - n^{(ab)}_{\hat{\Theta}^2,\Omega\Theta^2}\,\bar{\epsilon}\rho_{11} - n^{(ab)}_{\hat{\Theta},\Omega\Theta^2}\,\epsilon\rho_{12} \\ &+ 3\rho_{20} - n^{(ab)}_{\hat{\Theta}^2,\Omega\Theta}\,\epsilon\rho_{21} - n^{(ab)}_{\hat{\Theta},\Omega\Theta}\,\bar{\epsilon}\rho_{22}\Big]\end{aligned} \tag{5.87}$$

The lattice contributions are[9]

$$\begin{aligned}\tilde{\Lambda}^{\mathbf{A}} &= \frac{\sqrt{3}}{\eta^2}\sum_{m_1,m_2} e^{-3\pi l\left[m_2^2+\frac{4}{3}\left(\frac{m_1}{2}-m_2\right)^2\right]} = \sqrt{3}\chi_0 \\ \tilde{\Lambda}^{\mathbf{B}} &= \frac{1}{\sqrt{3}\eta^2}\sum_{m_1,m_2} e^{-\pi l\left[\frac{1}{3}m_1^2+4\left(\frac{m_1}{2}-m_2\right)^2\right]} = \frac{1}{\sqrt{3}}\sum_{i=0}^{2}\chi_i\end{aligned} \tag{5.88}$$

In order to cancel these tadpoles we now introduce D-branes with mixed boundary conditions. For both the **A** and the **B** lattice we choose three kinds of D-branes with $\theta \in \{\pi/2, \pi/6, -\pi/6\}$. The asymmetric $\widehat{\mathbb{Z}}_3$ cyclically permutes these three branes, whereas the symmetric $\mathbb{Z}_3$ leaves every brane invariant and acts with a $\gamma_{\Theta,i}$ matrix on the Chan-Paton factors of each brane. Since $\widehat{\mathbb{Z}}_3$ permutes the branes, all three $\gamma_{\Theta,i}$ actions must be the same. The computation of the annulus amplitude gives

$$\begin{aligned}A^{(ab)} = \\ \frac{2c}{12}\int_0^\infty \frac{dt}{t^4}\frac{1}{\eta^4}\Big[&\mathrm{M}^2\rho_{00}\,\Lambda^a\Lambda^b + (\mathrm{Tr}\,\gamma_\Theta)^2\,\rho_{01} + (\mathrm{Tr}\,\gamma_{\Theta^2})^2\,\rho_{02}+ \\ &\mathrm{M}^2\,n^{(ab)}_{\hat{\Theta},1}\,\rho_{10} + (\mathrm{Tr}\,\gamma_\Theta)^2\,n^{(ab)}_{\hat{\Theta},\Theta}\,\epsilon\rho_{11} + (\mathrm{Tr}\,\gamma_{\Theta^2})^2\,n^{(ab)}_{\hat{\Theta},\Theta^2}\,\bar{\epsilon}\rho_{12}+ \\ &\mathrm{M}^2\,n^{(ab)}_{\hat{\Theta}^2,1}\,\rho_{20} + (\mathrm{Tr}\,\gamma_\Theta)^2\,n^{(ab)}_{\hat{\Theta}^2,\Theta}\,\bar{\epsilon}\rho_{21} + (\mathrm{Tr}\,\gamma_{\Theta^2})^2\,n^{(ab)}_{\hat{\Theta}^2,\Theta^2}\,\epsilon\rho_{22}\Big]\end{aligned} \tag{5.89}$$

where the $\hat{\theta}$ twisted sector is given by open strings stretched between D-branes with θ_i and θ_{i+1}. Thus, $n^{(ab)}_{\hat{\Theta},1}$ denotes the intersection number of two such branes and $n^{(ab)}_{\hat{\Theta},\Theta}$ the number of intersection points invariant under Θ. The actual numbers turn out to be the same as the multiplicities of the closed string twisted sectors in (5.85) and (5.86). For the tree channel amplitude we obtain

$$\begin{aligned}\tilde{A}^{(ab)} = 2\frac{c}{6}\int_0^\infty dl\frac{1}{\eta^4}\Big[&\mathrm{M}^2\left(\rho_{00}\,\tilde{\Lambda}^a\tilde{\Lambda}^b + \frac{1}{3}n^{(ab)}_{\hat{\Theta}^2,1}\,\rho_{01} + \frac{1}{3}n^{(ab)}_{\hat{\Theta},1}\,\rho_{02}\right) \\ &+ (\mathrm{Tr}\,\gamma_\Theta)^2\left(3\rho_{10} - n^{(ab)}_{\hat{\Theta}^2,\Theta}\,\bar{\epsilon}\rho_{11} - n^{(ab)}_{\hat{\Theta},\Theta}\,\epsilon\rho_{12}\right) \\ &+ (\mathrm{Tr}\,\gamma_{\Theta^2})^2\left(3\rho_{20} - n^{(ab)}_{\hat{\Theta}^2,\Theta^2}\,\epsilon\rho_{21} - n^{(ab)}_{\hat{\Theta},\Theta^2}\,\bar{\epsilon}\rho_{22}\right)\Big]\end{aligned} \tag{5.90}$$

[9] The argument of the $SU(3)$ characters is $q = \exp(-4\pi l)$.

ϵ	(ab)	spectrum
1	–	$(1,1)\,\text{Sugra} + 4\times V_{1,1}$
$e^{\pm 2\pi i/3}$	**AA**	$(0,1)\,\text{Sugra} + 6\times T + 15\times H$
	AB	$(0,1)\,\text{Sugra} + 9\times T + 12\times H$
	BB	$(0,1)\,\text{Sugra} + 10\times T + 11\times H$

Table 5.1: Closed string spectra of the $\left(T^2\times T^2\right)/\left(\mathbb{Z}_3^L\times\mathbb{Z}_3^R\right)$-orientifold

Finally, one has to compute the Möbius amplitude

$$\begin{aligned} &M^{(ab)} = \\ &-\frac{2c}{12}\int_0^\infty \frac{dt}{t^4}\frac{1}{\eta^4}\Big[\mathrm{M}\,\rho_{00}\,\Lambda^a\Lambda^b + \mathrm{Tr}(\gamma_{\Omega\Theta}^T\gamma_{\Omega\Theta}^{-1})\,\rho_{01} + \mathrm{Tr}(\gamma_{\Omega\Theta^2}^T\gamma_{\Omega\Theta^2}^{-1})\,\rho_{02} \\ &\qquad +\mathrm{M}\,n^{(ab)}_{\hat\Theta,\Omega}\,\rho_{11} + \mathrm{Tr}(\gamma_{\Omega\Theta}^T\gamma_{\Omega\Theta}^{-1})\,n^{(ab)}_{\hat\Theta,\Omega\Theta}\,\epsilon\rho_{12} + \mathrm{Tr}(\gamma_{\Omega\Theta^2}^T\gamma_{\Omega\Theta^2}^{-1})\,n^{(ab)}_{\hat\Theta,\Omega\Theta^2}\,\bar\epsilon\rho_{10} \\ &\qquad +\mathrm{M}\,n^{(ab)}_{\hat\Theta^2,\Omega}\rho_{22} + \mathrm{Tr}(\gamma_{\Omega\Theta}^T\gamma_{\Omega\Theta}^{-1})\,n^{(ab)}_{\hat\Theta^2,\Omega\Theta}\,\bar\epsilon\rho_{20} + \mathrm{Tr}(\gamma_{\Omega\Theta^2}^T\gamma_{\Omega\Theta^2}^{-1})n^{(ab)}_{\hat\Theta^2,\Omega\Theta^2}\epsilon\rho_{21}\Big] \end{aligned} \tag{5.91}$$

with argument $q=-\exp(-2\pi t)$. Transformation into tree channel leads to the expression

$$\begin{aligned} \widetilde{M}^{(ab)} = -2\frac{8c}{3}\int_0^\infty dl\frac{1}{\eta^4}\Big[&\mathrm{M}\left(\rho_{00}\,\tilde\Lambda^a\tilde\Lambda^b + \frac{1}{3}n^{(ab)}_{\hat\Theta,\Omega}\,\rho_{01} + \frac{1}{3}n^{(ab)}_{\hat\Theta^2,\Omega}\,\rho_{02}\right) \\ &+ \mathrm{Tr}(\gamma_{\Omega\Theta^2}^T\gamma_{\Omega\Theta^2}^{-1})\left(3\rho_{11} - n^{(ab)}_{\hat\Theta,\Omega\Theta^2}\,\bar\epsilon\rho_{12} - n^{(ab)}_{\hat\Theta^2,\Omega\Theta^2}\,\epsilon\rho_{10}\right) \\ &+ \mathrm{Tr}(\gamma_{\Omega\Theta}^T\gamma_{\Omega\Theta}^{-1})\left(3\rho_{22} - n^{(ab)}_{\hat\Theta,\Omega\Theta}\,\epsilon\rho_{20} - n^{(ab)}_{\hat\Theta^2,\Omega\Theta}\,\bar\epsilon\rho_{21}\right)\Big] \end{aligned} \tag{5.92}$$

The three tree channel amplitudes give rise to two tadpole cancellation conditions

$$\begin{aligned} &\mathrm{M}^2 - 16\,\mathrm{M} + 64 = 0, \\ &(\mathrm{Tr}\,\gamma_\Theta)^2 - 16\mathrm{Tr}(\gamma_{\Omega\Theta^2}^T\gamma_{\Omega\Theta^2}^{-1}) + 64 = 0 \end{aligned} \tag{5.93}$$

Thus, we have $\mathrm{M}=8$ D9-branes of each kind and the action of $\mathbb{Z}_3$ on the Chan-Paton labels has to satisfy $\mathrm{Tr}\gamma_\Theta = 8$ implying that we have the simple solution that γ_Θ is the identity matrix.

5.4.1.2 The massless spectrum

Having solved the tadpole cancellation conditions we can move forward and compute the massless spectrum of the effective commutative field theory in the non-compact six-dimensional space-time. In computing the massless spectra we have to take into account the actions of the operations on the various fixed points. In the closed string sector we find the spectra shown in table 5.1. The computation of the massless spectra in the open string sector is also straightforward and yields the result in table 5.2.

ϵ	(ab)	spectrum
1	–	$V_{1,1}$ in $SO(8)$
$e^{\pm 2\pi i/3}$	**AA**	V in $SO(8)$ $+4\times H$ in **28**
	AB	V in $SO(8)$ $+1\times H$ in **28**
	BB	V in $SO(8)$

Table 5.2: Open string spectra of the $\left(T^2 \times T^2\right)/\left(\mathbb{Z}_3^L \times \mathbb{Z}_3^R\right)$-orientifold

All the spectra shown in table 5.1 and table 5.2 satisfy the cancellation of the non-factorizable anomaly. Note, that the configurations **AB** and **BB** were not analyzed in [53]. Thus, we have successfully applied the techniques derived in section 5.2 and section 5.3 to the construction of asymmetric orientifolds.

Concluding remarks

In this chapter we have pointed out a relationship between the realization of asymmetric operations in the open string sector and non-commutative geometry arising at the boundaries of open string world-sheets. More concretely, we have shown that a left-right asymmetric rotation transforms an ordinary Neumann or Dirichlet boundary condition into a mixed Neumann-Dirichlet boundary condition. We have employed this observation to rederive the non-commutativity relations for the open string. Moreover, we have solved the problem of how the open string sector manages to incorporate asymmetric symmetries. It simply turns on background gauge fluxes. Finally, we have considered a concrete asymmetric Type I vacuum, where D-branes with mixed boundary conditions were introduced to cancel all tadpoles.

We have restricted ourselves to the case of products of two-dimensional tori. With the insights gained in the preceeding chapter it is very suggestive how to generalize these ideas to more general asymmetric elements of the T-duality group. As it is known that the T-duality group $SO(d,d,\mathbb{Z})$ is generated by only three classes of generators,[10] we can map each kind of constant open string boundary condition under T-duality in addtion to the closed string background fields.

It would be also interisting to discuss the dual heterotic description.

Furthermore, it would be interesting to see whether via the asymmetric rotation one can gain further insight into the relation between the effective non-commutative and commutative gauge theories on the branes.

[10] I.e. integer shifts in the NSNS B-field, change of the torus-basis ($\in SL(d,\mathbb{Z})$) and a so called factorized duality, which is the generalization of the $R\alpha' \to \alpha'/R$ T-duality. A nice review on T-duality is [34]. There also the original references (which are quite a lot) concerning the generators of $SO(d,d,\mathbb{Z})$ can be found.

Chapter 6

Toroidal orientifolds with magnetized versus intersecting D-Branes

In this chapter we will mainly present the results published in publication [2, 124]. We investigated strings on toroidal orientifolds with D9-branes which are allowed to carry arbitrary magnetic background fluxes. We restricted ourselves to block diagonal NS $U(1)$ fields F. Therefore the pure Ω-orientifold is T-dual to a $\bar{\sigma}\Omega$-orientifold with $\bar{\sigma}$ being a reflection which has a fixed point locus of half the dimension of the compact space. As a consequence the T-dual picture leads to an orientifold 6-plane (O6-plane) in compactifications on six-tori (and to an O7-plane in compactifications on four-tori). The O6-plane is charged under a RR 7-form while the O7-plane carries RR 8-form charge. The RR charge will be canceled by D6- (resp. D7-) branes without any $U(1)$ background field. The T-dual picture has the advantage to admit a purely geometric interpretation, with multiplicities of open string states given by the intersection numbers of the corresponding Dp branes. Because the interpretation of the intersection numbers in the F-field picture is less direct though possible (cf. section 4.2.1.3, p. 100), we will from time to time switch between the description in terms of fluxes and the purely geometrical description in terms of intersecting Dp branes. We will discuss both the technical description as well as applications to phenomenology.

6.1 Introduction

The search for realistic string vacua is one of the burning open problems within superstring theory. A phenomenologically viable string compactification should contain at least three chiral fermion generations, the Standard Model gauge group and broken space-time supersymmetry. In the context of 'conventional' string compactifications the requirement of getting chiral fermions is usually achieved by considering compact, internal background spaces with nontrivial topology rather than simple tori. In particular, when analyzing the Kaluza-Klein fermion spectra [125] a net-fermion generation number arises if the internal Dirac operator has zero modes. For example, considering heterotic string compactifications on Calabi-Yau threefolds [47] , the net-generation number is equal to $|\chi|/2$, where χ is the Euler number of the Calabi-Yau space. Chiral fermions are also present in a large class of heterotic orbifold compactifications [32], as well as in free bosonic [126] and fermionic [127, 128] constructions. Type II string models with chiral fermions can be constructed by locating D-branes at transversal orbifold or conifold singularities [29, 30], or by considering intersections of D-branes and NS-branes [129, 130, 131]; chiral Type I models were first proposed in [132]. Moreover orbifold compactifications of eleven-dimensional M-theory can lead to chiral fermions, as discussed e.g. in [133, 134, 135, 136, 137].

The phenomenological requirement of breaking space-time supersymmetry can be met in various ways. In the context of heterotic string compactifications gaugino condensation [138, 139] or the Scherk-Schwarz mechanism [46, 140, 141, 142, 143, 144] lead to potentially interesting models with supersymmetry broken at low energies. In addition, as it was realized more recently, Type II models on nontrivial background spaces with certain D-brane configurations possess broken space-time supersymmetry. Especially, when changing the GSO-projections tachyon free Type 0 orientifolds in four dimensions can be constructed [145, 146, 147, 148]. Alternatively, orientifolds on six-dimensional orbifolds with brane-antibrane configurations provide interesting scenarios [149, 150, 151, 152, 153, 154], where supersymmetry is left unbroken in the gravity bulk, but broken in the open string sector living on the brane-antibrane system.

Finally the quest for a realistic gauge group with sufficiently low rank is met in heterotic strings by choosing appropriate gauge vector bundles on the Calabi-Yau spaces [155], which can be alternatively described by turning on *Wilson lines* in Calabi-Yau or also in orbifold compactifications [156]. [1] On the Type II side the gauge group can be reduced by Wilson lines or, in the T-dual picture, by placing the branes at different positions inside the internal space.

As it should have become clear from the previous discussion, 'standard' heterotic, Type I or Type II compactifications on simple 6-tori do not meet any of the three above requirements. However, as we will discuss in 6.2.1, turning on magnetic fluxes in the internal directions of the D-branes, thereby inducing mixed Neumann-Dirichlet boundary conditions for open strings equivalent to a non-commutative internal geometry [104, 157] on the branes, all three goals can be achieved in one single stroke.[2] Specifically, we will discuss Type I string compactifications on a product of d non-commutative two-tori to $10-2d$ non-compact Minkowski dimensions ($d = 2, 3$), i.e. the ten-dimensional background

[1] A Wilson line defines an embedding of the fundamental group $\pi_1(\mathcal{X})$ into the gauge bundle of the theory. Wilson lines on compactification spaces $\mathcal{X}$ with abelian fundamental group only lead to rank conserving symmetry breaking. However in manifolds with *nonabelian* $\pi_1(\mathcal{X})$ one can achieve rank reducing gauge-symmetry breaking via Wilson lines.

[2] In chapter 5 which is mainly based on [1] we have discussed Type I string compactifications on non-commutative asymmetric orbifold spaces.

spaces $\mathcal{M}_{10}$ we are considering have the following form:

$$\mathcal{M}_{10} = \mathbb{R}^{1,9-2d} \times \mathcal{X}_{2d}, \qquad \mathcal{X}_{2d} = \prod_{j=1}^{d} T^2_{(j)} \tag{6.1}$$

Since we assume that the purely internal magnetic F-field is block diagonal and constant for all D9-branes:

$$F_{ab} = \bigoplus_{j=1}^{d} \begin{pmatrix} 0 & F^{(j)} \\ -F^{(j)} & 0 \end{pmatrix} \tag{6.2}$$

these boundary conditions are T-dual to D$(9-d)$-branes which have an angle of

$$\phi^{(j)} = \arctan(F^{(j)} + B^{(j)}) \tag{6.3}$$

wrt. the $X^{(j)}$ axis. (T-duality is performed along the Y-direction: $R_Y/\sqrt{\alpha'} \to \sqrt{\alpha'} R_Y$.) Applying the results of chapter 4 we find that the open string coordinates of the D9-branes fulfill the following equal-τ commutation relation in the F (= *flux*) -picture:

$$[X_{10-2j}(\tau,\sigma), X_{11-2j}(\tau,\sigma)]\big|_{\sigma\in\partial\mathcal{M}} = i\Theta^{(j)}\,, \qquad j = 1,\ldots,d \tag{6.4}$$

The non-commutative deformation parameter $\Theta^{(j)}$ in eq. (6.4) is defined by:

$$\Theta^{(j)} \equiv -2\pi\alpha' \frac{F^{(j)} + B^{(j)}}{1 + (F^{(j)} + B^{(j)})^2} \tag{6.5}$$

The entire internal non-commutative torus will actually consist out of different sectors with different non-commutative deformation parameters, because we will introduce several D9-branes with different magnetic fluxes. We will show that the spectrum of open strings, with mixed boundary conditions in the internal directions is generically chiral, breaks space-time supersymmetry and leads to gauge groups of lower rank. It is however important to stress that the effective gauge theories in the uncompactified part of space-time are still commutative, and therefore are Lorentz invariant and local field theories.

This construction is the D-brane extended version of [28], where it was already observed that turning on magnetic flux in a toroidal Type I compactification leads to supersymmetry breaking and chiral massless spectra in four space-time dimensions. However, the consistency conditions for such models were derived in the effective non-supersymmetric gauge theories, leaving the actual string theoretic conditions an open issue. We will show that, with all the insights gained in the description of D-branes with magnetic flux, we are now able to achieve a complete string theoretic understanding, giving rise to certain extensions and modifications of the purely field theoretical analysis. As a solution to the tadpole cancellation conditions we can get different sectors of D-branes with different magnetic fluxes, corresponding to different non-commutative boundary conditions. Chirality then arises in sectors of open strings which have ends on branes with different gauge flux, while the presence of any solitary flux is not sufficient. The gauge groups that act on the

D-branes with non-vanishing flux are unitary instead of orthogonal or symplectic in accord with the general statement that only these are compatible with a non-commutative deformation of the coordinate algebra.

As already mentioned, it is sometimes very helpful to employ an equivalent T-dual description, where the background fields vanish and the torus is entirely commutative, but the D$(9-d)$-branes intersect at various different angles [56, 57, 58]. This description allows to present a more intuitive picture of the open string sector involved in such models. Chiral fermions then arise due to the nontrivial geometric boundary conditions of the intersecting D-branes,[3] which at the same time generically break space-time supersymmetry [4] and lower the rank of the gauge group.

The chapter is organized as follows. In the next section we analyze the one-loop amplitudes and the resulting tadpole cancellation conditions for D9-branes with mixed Neumann-Dirichlet boundary conditions moving in the background of d two-dimensional tori ($d = 2, 3$). In section 6.3 we discuss specific six-dimensional models ($d = 2$) working out the non-supersymmetric, chiral spectrum. We also point out some subtleties involving the mechanisms of supersymmetry breaking in 'nearly' supersymmetric brane configurations. In section 6.4 we move on to chiral, non-supersymmetric four-dimensional models ($d = 3$), reconsider in particular the model presented in [28] with GUT-like gauge group $G = U(5) \times U(3) \times U(4) \times U(4)$ and display another 4 generation model with 'Standard Model' gauge group $G = U(3) \times U(2) \times U(1)^r$.[5] Some phenomenological problems of this model are stressed at the end. This chapter is organized as follows. In the next section we analyze the one-loop amplitudes and the resulting tadpole cancellation conditions for D9-branes with mixed Neumann-Dirichlet boundary conditions moving in the background of d two-dimensional tori ($d = 2, 3$).

6.2 One loop amplitudes

In [28] it was observed that turning on magnetic flux in a toroidal Type I compactification leads to supersymmetry breaking and in general to chiral massless spectra in four space-time dimensions. The consistency conditions for such models were derived in the effective non-supersymmetric gauge theory but not in the full string theory. In this section we will show that, with the inclusion of D-branes with magnetic flux, respectively D-branes at angles, we are now able to derive the string theoretic tadpole cancellation conditions.

[3] The appearance of *chiral* fermions at intersections of angled D-branes was discovered in [27].

[4] However if the angles $\Delta\phi_j$ of the branes fulfill special conditions e.g. $\sum_{j=1}^{d} \Delta\phi_j = 0$ supersymmetry is preserved. It will turn out that requiring RR-tadpole cancellation in purely toroidal $\bar{\sigma}\Omega$ orientifolds has only supersymmetric solutions with $\phi_j = 0 \ \forall j$. This excludes chirality.

[5] For other recent bottom up attempts to obtain GUTs and the Standard Model from branes see [158, 159, 160].

6.2.1 D9-branes with magnetic fluxes

As our starting point we consider the orientifold

$$\frac{\text{Type IIB on } T^{2d}}{\Omega} \tag{6.6}$$

In the following we will assume that T^{2d} splits into a direct product of d two-dimensional tori $T^2_{(j)}$ with coordinates $X_1^{(j)}$, $X_2^{(j)}$ and radii $R_1^{(j)}$, $R_2^{(j)}$, $j = 1, \ldots, d$. We restrict ourselves to purely imaginary complex structures and vanishing antisymmetric NSNS tensor field B.[6] Turning on magnetic flux F_{ab} on a D9-brane changes the pure Neumann boundary conditions into mixed Neumann-Dirichlet conditions. This case was investigated in the preceding chapter(s). Especially we gave formulæ for the masses of the momentum modes (eq. (5.36), p. 126). In addition we gave a formula for the zero mode degeneracy (eq. (5.37)). In chapter 4 we related the modings of the oscillator modes to the Eigenvalues of a matrix constructed form the two $U(1)$ fields coupling to the endpoints of the string. However here we will use the "classical" derivation of the solution given by [105]. The boundary conditions for strings with F-field are given by (cf. (5.73), p. 132):

$$\begin{aligned} \partial_\sigma X_1 + \cot\phi_i\, \partial_\tau X_2 &= 0 \\ \partial_\sigma X_2 - \cot\phi_i\, \partial_\tau X_1 &= 0 \end{aligned} \quad \text{with} \quad \begin{cases} i = 1 & \text{and } \sigma = 0 \\ i = 2 & \text{and } \sigma = \pi \end{cases} \tag{6.7}$$

The angles ϕ_i are related to the F-fields that couple to the string via:

$$\mathcal{F}_1 = \begin{pmatrix} 0 & -\cot\phi_1 \\ \cot\phi_1 & 0 \end{pmatrix} \qquad \mathcal{F}_2 = \begin{pmatrix} 0 & \cot\phi_2 \\ -\cot\phi_2 & 0 \end{pmatrix} \tag{6.8}$$

The mode expansion that solves these boundary conditions is

$$\begin{aligned} X_1 = x_1 - &\sqrt{\alpha'} \sum_{n\in\mathbb{Z}} \frac{\alpha_{n+\nu}}{n+\nu} e^{-i(n+\nu)\tau} \sin[(n+\nu)\sigma + \phi_1] - \\ &\sqrt{\alpha'} \sum_{m\in\mathbb{Z}} \frac{\alpha_{m-\nu}}{m-\nu} e^{-i(m-\nu)\tau} \sin[(m-\nu)\sigma - \phi_1], \\ X_2 = x_2 + &i\sqrt{\alpha'} \sum_{n\in\mathbb{Z}} \frac{\alpha_{n+\nu}}{n+\nu} e^{-i(n+\nu)\tau} \sin[(n+\nu)\sigma + \phi_1] - \\ &i\sqrt{\alpha'} \sum_{m\in\mathbb{Z}} \frac{\alpha_{m-\nu}}{m-\nu} e^{-i(m-\nu)\tau} \sin[(m-\nu)\sigma - \phi_1] \end{aligned} \tag{6.9}$$

We will not review the mass formulæ. Instead of working with D9-branes with various magnetic fluxes, we will now use the T-dual description in terms of D-branes at angles [56, 57, 58], which allows to present a more intuitive picture of the open string sector involved in such models:

[6] As noted in chapter 2.2, eq. (2.40) a quantized B-field would allow Ω to be a symmetry. The case of non-vanishing NS B-fields on some T^2 and its T-dual interpretation was considered in [161].

6.2.2 D$(9-d)$-branes at angles

Applying a T-duality in all $X_2^{(j)}$ directions (which is a special version of the D-duality (2.67), p. 38)

$$R_2^{(j)} \rightarrow R_2^{(j)\prime} = 1/R_2^{(j)}$$

leads to boundary conditions for D$(9-d)$-branes intersecting at angles, where the angle of the D$(9-d)$-brane relative to the $X_1^{(j)}$ axes is given by

$$\tan\phi^{(j)} = F^{(j)} \tag{6.10}$$

(In the following we will omit the prime on the dual radii.) This T-duality also maps Ω onto $\bar{\sigma}\Omega$, where $\bar{\sigma}$ acts as complex conjugation on all the d complex coordinates along the $T^2_{(j)}$ tori. Thus, instead of (6.6) we are considering the orientifold

$$\frac{\text{Type II on } T^{2d}}{\bar{\sigma}\Omega} \tag{6.11}$$

For d even we have to take Type IIB and for d odd Type IIA, as explained in section 2.2.2.2. Note that after performing this T-duality transformation the internal coordinates are completely commutative.

Let $j \in \{1,\dots,d\}$ again label the d different two-dimensional tori and $a \in \{1,\dots,K\}$ the different kinds of D$(9-d)$-branes, which are distinguished by different angles on at least one torus. Moreover, we are only considering branes which do not densely cover any of the two-dimensional tori. Thus, the position of a D$(9-d)$-brane is described by two sets of integers $(n_a^{(j)}, m_a^{(j)})$, labeling how often the D-branes are wound around the two fundamental cycles of each $T^2_{(j)}$. The angles of such a brane with the axes $X_1^{(j)}$ are given by (cf. figure 5.2, p.123, were we assumed the lattice vector e_1 to be parallel to the $\mathfrak{Re}$-axis.):

$$\cot(\phi^{(j)}) = \frac{n^{(j)} + m^{(j)}\tau_1^{(j)}}{m^{(j)}\tau_2^{(j)}} = \cot(\alpha) + \frac{n_a^{(j)} R_1^{(j)}}{m^{(j)}\sin(\alpha) R_2^{(j)}} \tag{6.12}$$

α is the angle between the two generating lattice vectors: $\mathfrak{Re}(\tau) = \cos(\alpha) R_2^{(j)}$, $\mathfrak{Im}(\tau) = \sin(\alpha) R_2^{(j)}$. These conventions are shown in figure 5.2, p. 123, where we have omitted the index (j) which labels the different two-tori $T^2_{(j)}$. However in order for $\bar{\sigma}$ to be a symmetry the complex structure is fixed either to be purely imaginary or to have real part $\tau_1 = 1/2$ (cf. section 2.2.2.2, p. 40). The open string boundary condition is easily derived:

$$\partial_- X(\tau,\sigma) = \mathcal{R}(\vec{\phi})\partial_+ X(\tau,\sigma)\big|_{\sigma\in\partial\mathcal{M}} \qquad \mathcal{R} = D(\vec{\phi})^T \bar{\sigma} D(\vec{\phi}) \tag{6.13}$$

$D(\vec{\phi})$ is a rotation described by a set of angles $\vec{\phi} = (\phi^{(1)},\dots,\phi^{(d)})$. In our case $D(\vec{\phi})$ is block-diagonal, each 2×2 block acting on a $T^2_{(j)}$. In all cases considered in this thesis $\mathcal{R}$ is also block diagonal (except for chapter 4, where our considerations are more general). In orthogonal coordinates each block is of the form:

$$\mathcal{R}^{(j)}(\phi^{(j)}) = \begin{pmatrix} \cos(2\,\phi^{(j)}) & -\sin(2\,\phi^{(j)}) \\ -\sin(2\,\phi^{(j)}) & -\cos(2\,\phi^{(j)}) \end{pmatrix} \tag{6.14}$$

In complex coordinates (i.e. $Z = 1/\sqrt{2}(X_1 + iX_2)$) $\mathcal{R}^{(j)}\big(\phi^{(j)}\big)$ this combination of a reflection and a subsequent rotation by $-2\,\phi^{(j)}$ looks very simple:

$$\mathcal{R}^{(j)}\big(\phi^{(j)}\big)\big(Z^{(j)}\big) = e^{-2\phi^{(j)}}\bar{Z}^{(j)} \qquad \mathcal{R}^{(j)}\big(\phi^{(j)}\big)\big(\bar{Z}^{(j)}\big) = e^{2\phi^{(j)}}Z^{(j)} \tag{6.15}$$

Since $\bar{\sigma}\Omega$ reflects the D-branes at the axis $X_1^{(j)}$, for each brane labeled by $\big(n_a^{(j)}, m_a^{(j)}\big)$ we must also introduce the mirror brane with $\big(n_{a'}^{(j)}, m_{a'}^{(j)}\big) = \big(n_a^{(j)}, -m_a^{(j)}\big)$. The values $m_a^{(j)} = 0 \neq n_a^{(j)}$ and $n_a^{(j)} = 0 \neq m_a^{(j)}$ correspond to branes located along one of the axis. The horizontal D-branes translate via T-duality into D9-branes with vanishing flux and the vertical ones into branes of lower dimension with pure Dirichlet boundary conditions. A solution to the boundary conditions (6.13) was given in section 5.2.4.2, p. 128 and section 5.3.3, p. 131.

The questions we are going to deal with in the following are: Is it possible to cancel all or at least the RR tadpoles originating from the Klein bottle amplitude by D9-branes with non-vanishing magnetic fluxes $F^{(j)}$, or equivalently by D$(9-d)$-branes at nontrivial angles $\phi^{(j)}$? Taken that supersymmetry is broken generically by such a background, are there configuration which still preserve some amount of supersymmetry? This would provide a string scenario with partial supersymmetry breaking. Finally, what are the phenomenological properties of such compactifications? Concerning the first question we find a positive answer in the sense that the RR tadpole can be canceled, while supersymmetry is always broken entirely. This shows up both in the spectrum and a non-vanishing NSNS tadpole. Tachyons are always present in compactifications on T^4 and for some region in parameter space of the four dimensional compactifications as well. However it was shown after our publication appeared ([162, 163, 164]) that there exist toroidal models, which are free from tachyons (but still suffering from NSNS tadpoles). Interestingly, models with non-trivially intersecting D-branes generically contain chiral fermions motivating us to study how far one can get in deriving the Standard Model in this setting. However, later we will mention an obstacle to construct phenomenologically realistic models in this simple approach on a $T^6 = (T^2)^3$ which is a product of only **A**-type two-tori.

Technically we first have to compute all contributions to the massless RR tadpole. The cancellation conditions will then imply relations for the number of D9-branes and their respective background fluxes. This computation will be performed in the T-dual picture, where D9-branes with background fields are mapped to D$(9-d)$-branes, and the background fields translate into relative angles. This picture allows to visualize the D-branes easily and gives a much better intuition than dealing with sets of D9-branes, all filling the same space but differing by background fields.

6.2.3 Klein bottle amplitude

The loop channel Klein bottle amplitude for (6.11) can be computed straightforwardly

$$K = 2^{(5-d)} c\,(1-1) \int_0^\infty \frac{dt}{t^{(6-d)}} \frac{1}{4} \frac{\vartheta\begin{bmatrix}0\\1/2\end{bmatrix}^4}{\eta^{12}} \prod_{j=1}^{d} \left(\sum_{r,s\in\mathbb{Z}} e^{-\pi t\left(r^2/\left(R_1^{(j)}\right)^2 + s^2\left(R_2^{(j)}\right)^2\right)} \right) \tag{6.16}$$

with $c = V_{10-2d}/\left(8\pi^2\alpha'\right)^{5-d}$. Transforming (6.16) into tree channel, one obtains the following massless RR tadpole

$$\int_0^\infty dl\, 2^{(13-d)} \prod_{j=1}^{d} \left(\frac{R_1^{(j)}}{R_2^{(j)}} \right). \tag{6.17}$$

The tree channel Klein bottle amplitude allows to determine the normalization of the corresponding cross-cap states

$$|C\rangle = 2^{(d/2-4)} \left(\prod_{j=1}^{d} \frac{R_1^{(j)}}{R_2^{(j)}} \right)^{\frac{1}{2}} \left(|C_{\rm NS}\rangle + |C_{\rm R}\rangle\right) \tag{6.18}$$

6.2.4 Annulus amplitude

Next we calculate all contributions of open strings stretching between the various D$(9-d)$-branes, generically located at nontrivial relative angles. We will both include the case, where the relative angle is vanishing, i.e. the background gauge flux is equal on both branes, and the case, where the angle is $\pi/2$ and the field gets infinitely large on, say, p of the tori.

We start with the contributions of strings with both ends on the same brane. The T-dual of the Kaluza-Klein and winding spectrum in eq. (5.34) (p. 126) is given by (5.55) and (5.55) (p. 129). With our simplifying assumptions that $B=0$ it reads:[7]

$$M_a^2 = \sum_{j=1}^{d} \left(\left(\frac{r_a^{(j)}}{V_a^{(j)}} \right)^2 + \left(s_a^{(j)}\right)^2 \left(\frac{R_1^{(j)} R_2^{(j)}}{V_a^{(j)}} \right)^2 \right) \tag{6.19}$$

with

$$V_a^{(j)} = \sqrt{\left(R_1^{(j)} n_a^{(j)}\right)^2 + \left(R_2^{(j)} m_a^{(j)}\right)^2} \tag{6.20}$$

denoting the volume of the brane on $T^2_{(j)}$. It is now straightforward to compute the loop channel annulus amplitude for open strings starting and ending on the same brane and transform it to the tree channel

$$\tilde{A}_{aa} = c\, N_a^2 (1-1) \int_0^\infty dl\, \frac{1}{2^{(d+1)}} \prod_{j=1}^{d} \frac{\left(V_a^{(j)}\right)^2}{R_1^{(j)} R_2^{(j)}} \, \frac{\vartheta\begin{bmatrix}1/2\\0\end{bmatrix}^4}{\eta^{12}} \sum_{r,s} e^{-\pi l \widetilde{M}_a^2} \tag{6.21}$$

[7] $B \neq 0$ does not modify the the RR tadpole conditions, as they are topological. It also does not change the moding of the oscillators, as it does not enter the boundary conditions. Therefore $B=0$ is a very mild simplification.

with

$$\widetilde{M}_a^2 = \sum_{j=1}^{d} \left(\left(r_a^{(j)}\right)^2 \left(V_a^{(j)}\right)^2 + \left(s_a^{(j)}\right)^2 \left(\frac{V_a^{(j)}}{R_1^{(j)} R_2^{(j)}} \right)^2 \right) \tag{6.22}$$

N_a counts the numbers of different kinds of branes. Using (6.21) one can determine the normalization of the boundary state, which has the schematic form

$$|\, D_a \rangle = 2^{-(d/2+1)} \left(\prod_{j=1}^{d} \frac{V_a^{(j)}}{\sqrt{R_1^{(j)} R_2^{(j)}}} \right) \left(|\, D_{a,\mathrm{NS}} \rangle + |\, D_{a,\mathrm{R}} \rangle \right) \tag{6.23}$$

Reflecting the brane on a single $T^2_{(j)}$ by a π rotation onto itself corresponds to $(n_a^{(j)}, m_a^{(j)}) \to (-n_a^{(j)}, -m_a^{(j)})$ and, as can be determined in the boundary state approach, changes the sign of the RR charge, thus exchanging branes and anti-branes.

Using the boundary state (6.23) we can compute the tree channel annulus amplitude for an open string stretched between two different D-branes

$$\begin{aligned} \tilde{A}_{ab} &= \int_0^\infty dl\, \langle D_a | e^{-l H_{cl}} | D_b \rangle \\ &= \frac{c}{2} N_a N_b I_{ab} \int_0^\infty dl (-1)^d \sum_{\substack{\alpha,\beta \\ \in \{0, \frac{1}{2}\}}} (-1)^{2(\alpha+\beta)} \frac{\vartheta\begin{bmatrix}\alpha\\ \beta\end{bmatrix}^{4-d} \prod_{j=1}^d \vartheta\begin{bmatrix}\alpha\\ \Delta(\phi^{(j)})_{ab}+\beta\end{bmatrix}}{\eta^{12-3d} \prod_{j=1}^d \vartheta\begin{bmatrix}1/2\\ \Delta(\phi^{(j)})_{ab}+1/2\end{bmatrix}} \end{aligned} \tag{6.24}$$

where the coefficient

$$I_{ab} = \prod_{j=1}^{d} \left(n_a^{(j)} m_b^{(j)} - m_a^{(j)} n_b^{(j)} \right) \tag{6.25}$$

is the (oriented) *intersection number* of the two branes. We have defined the oriented angle between brane a and b on the torus $T^2_{(j)}$ by:

$$\Delta(\phi^{(j)})_{ab} \equiv \left(\phi_b^{(j)} - \phi_a^{(j)}\right)/\pi \tag{6.26}$$

It gives rise to an extra multiplicity in the annulus loop channel, which we have to take into account, when we compute the massless spectrum. In order to properly include the case where some $\phi_a^{(j)} = \phi_b^{(j)}$, one needs to employ the relation

$$\lim_{\psi \to 0} \frac{2 \sin(\pi \psi)}{\vartheta\begin{bmatrix}1/2\\ 1/2+\psi\end{bmatrix}} = -\frac{1}{\eta^3} \tag{6.27}$$

and include a sum over KK momenta and windings as in (6.21). The contribution to the massless RR tadpole due to (6.21) and (6.24) is

$$\int_0^\infty dl\, N_a N_b\, 2^{(3-d)} \prod_{j=1}^{d} \frac{\left(R_1^{(j)}\right)^2 n_a^{(j)} n_b^{(j)} + \left(R_2^{(j)}\right)^2 m_a^{(j)} m_b^{(j)}}{R_1^{(j)} R_2^{(j)}} \tag{6.28}$$

The loop channel annulus can be obtained by a modular transformation:

$$A_{ab} = c\; N_a N_b\, I_{ab} \int_0^\infty \frac{dt}{t^{(6-d)}}\,\frac{1}{4}\cdot\frac{1}{\eta^{12-3d}}$$
$$\cdot \sum_{\alpha,\beta\in\{0,1/2\}} (-1)^{2(\alpha+\beta)}\,\frac{\vartheta\begin{bmatrix}\beta\\ \alpha\end{bmatrix}^{4-d}\;\prod_{j=1}^d e^{i2\pi(\frac{1}{2}-\alpha)\Delta(\phi^{(j)})_{ab}}\,\vartheta\begin{bmatrix}\Delta(\phi^{(j)})_{ab}+\beta\\ \alpha\end{bmatrix}}{\prod_{j=1}^d \vartheta\begin{bmatrix}\Delta(\phi^{(j)})_{ab}+1/2\\ 1/2\end{bmatrix}} \tag{6.29}$$

6.2.5 Möbius amplitude

Computing the overlap between the crosscap state (6.18) and a boundary state (6.23) yields the contribution of the brane $\mathrm{D}(9-p)_a$ to the Möbius amplitude

$$\widetilde{M}_a = \mp c\, N_a\, 2^5\, (-1)^d \int_0^\infty dl \prod_{j=1}^d m_a^{(j)}$$
$$\cdot \sum_{\alpha,\beta\in\{0,1/2\}} (-1)^{2(\alpha+\beta)}\,\frac{\vartheta\begin{bmatrix}\alpha\\ \beta\end{bmatrix}^{4-d}\;\prod_{j=1}^d \vartheta\begin{bmatrix}\alpha\\ \phi_a^{(j)}/\pi+\beta\end{bmatrix}}{\eta^{12-3d}\;\prod_{j=1}^d \vartheta\begin{bmatrix}1/2\\ \phi_a^{(j)}/\pi+1/2\end{bmatrix}} \tag{6.30}$$

with argument $q = -\exp(-4\pi l)$. Therefore the contribution to the RR tadpole is

$$\mp \int_0^\infty dl\, N_a\, 2^{(9-d)} \prod_{j=1}^d \left(\frac{R_1^{(j)}}{R_2^{(j)}} n_a^{(j)}\right) \tag{6.31}$$

The overall sign in (6.30) and (6.31) is fixed by the tadpole cancellation condition. In the loop channel the contribution of the Möbius strip results from strings starting on one brane and ending on its mirror partner. The extra multiplicity given by the numbers $m_a^{(j)}$ of intersection points invariant under $\bar\sigma$ needs to be regarded as before. Now we have all the ingredients to study the relations which derive from the cancellation of massless RR tadpoles.

6.3 Compactifications to six dimensions

We are compactifying Type I strings on a four-dimensional torus and cancel the tadpoles by introducing stacks of D9-branes with magnetic fluxes. The T-dual arrangement of D7-branes at angles looks like the situation depicted in figure 6.1, where we have drawn only two types of D7-branes labeled by a and b and their mirror partners a' and b', the angles being chosen arbitrary.

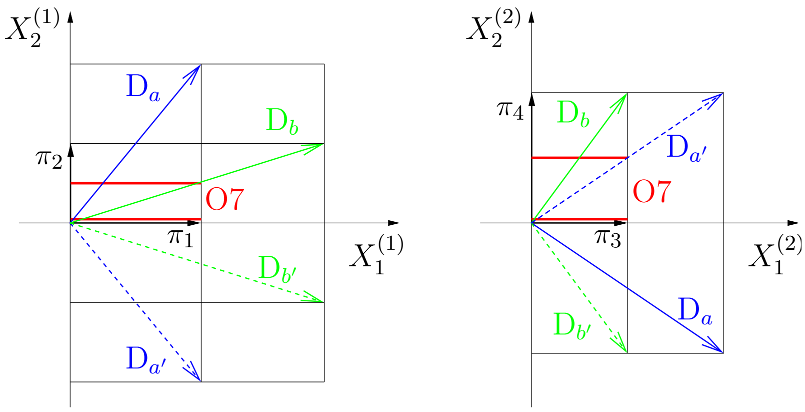

Figure 6.1: D7-brane configuration with $\bar{\sigma}$ images on the T^4. The orientifold 7 planes are painted in red. The fundamental one cycles of the torus are denoted by $\pi_1 \dots \pi_4$.

6.3.1 Six-dimensional models

The complete annulus amplitude is a sum over all open strings stretched between the various D7-branes

$$\begin{aligned}\tilde{A}_{\text{tot}} = \sum_{a=1}^{K} \left(\tilde{A}_{aa} + \tilde{A}_{a'a'} + \tilde{A}_{aa'} + \tilde{A}_{a'a}\right) + \\ \sum_{a<b} \left(\tilde{A}_{ab} + \tilde{A}_{ba} + \tilde{A}_{a'b'} + \tilde{A}_{b'a'} + \tilde{A}_{ab'} + \tilde{A}_{ba'} + \tilde{A}_{a'b} + \tilde{A}_{b'a}\right)\end{aligned} \quad (6.32)$$

Using (6.21) and (6.28) and adding up all these various contributions yields the following two RR tadpoles:

$$\int_0^\infty dl\, 8\, \frac{R_1^{(1)} R_1^{(2)}}{R_2^{(1)} R_2^{(2)}} \left(\sum_{a=1}^{K} N_a\, n_a^{(1)} n_a^{(2)}\right)^2 + \int_0^\infty dl\, 8\, \frac{R_2^{(1)} R_2^{(2)}}{R_1^{(1)} R_1^{(2)}} \left(\sum_{a=1}^{K} N_a\, m_a^{(1)} m_a^{(2)}\right)^2 \quad (6.33)$$

For the total Möbius amplitude we obtain the RR tadpole

$$\mp \int_0^\infty dl\, 2^8\, \frac{R_1^{(1)} R_1^{(2)}}{R_2^{(1)} R_2^{(2)}} \sum_{a=1}^{K} N_a\, n_a^{(1)} n_a^{(2)} \quad (6.34)$$

Note, the two special cases of N_9 horizontal and N_5 vertical D7-branes are contained in (6.33) and (6.34) by setting $N_a = N_9/2$ respectively $N_a = N_5/2$. Choosing the minus sign in (6.34) we get the two RR tadpole cancellation conditions

$$\begin{aligned}\left(\prod_{j=1}^{2} \tau_2^{(j)}\right)^{-1} = \frac{R_1^{(1)} R_1^{(2)}}{R_2^{(1)} R_2^{(2)}} : \qquad & \sum_{a=1}^{K} N_a\, n_a^{(1)} n_a^{(2)} = 16, \\ \prod_{j=1}^{2} \tau_2^{(j)} = \frac{R_2^{(1)} R_2^{(2)}}{R_1^{(1)} R_1^{(2)}} : \qquad & \sum_{a=1}^{K} N_a\, m_a^{(1)} m_a^{(2)} = 0\end{aligned} \quad (6.35)$$

As one might have expected, pure D9-branes with $m_a^{(j)} = 0$ only contribute to the tadpole proportional to the product of the inverse imaginary parts of the complex structures of the two-tori. The D5-branes with $n_a^{(j)} = 0$ are responsible only for the tadpole that is proportional to the product of the imaginary parts of the complex structures. Remarkably, by choosing multiple winding numbers, $n_a^{(j)} > 1$, one can reduce the rank of the gauge group. As usual in non-supersymmetric models, there remains an uncanceled NSNS tadpole, which needs to be canceled by a Fischler-Susskind mechanism. However it is also possible that the equations of motion lead to a degenerate compactification space (i.e. a degenerate torus).

In the section 6.5.1 we shall show that except for the trivial case, when $m_a^{(1)} = m_a^{(2)} = 0$ for all a, i.e. vanishing gauge flux on all the D9-branes, supersymmetry is broken and tachyons develop for open strings stretched between different branes. In contrast to the breaking of supersymmetry in a brane-antibrane system these tachyons cannot be removed by turning on Wilson-lines, which is related via T-duality to shifting the position of the branes by some constant vector. At any non trivial angle there always remains an intersection point of two D7-branes where the tachyons can localize. Also the lowest lying bosonic spectrum depends on the radii of the torus, which determine the relative angles. The zero point energy in the NS sector of a string stretching between two different branes is shifted by

$$\Delta E_{0,\mathrm{NS}} = \frac{1}{2} \sum_{j=1}^{d} \frac{\phi_a^{(j)} - \phi_b^{(j)}}{\pi} \tag{6.36}$$

using the convention $\phi_a^{(j)} - \phi_b^{(j)} \in (0, \pi/2]$. Even assuming a standard GSO projection, the lightest physical state can easily be seen to be tachyonic except for the supersymmetric situation with $\phi_a^{(1)} - \phi_b^{(1)} = \phi_a^{(2)} - \phi_b^{(2)}$. We shall find in the next section that tadpole cancellation prohibits this solution, except when all fluxes vanish.

On the contrary, the chiral fermionic massless spectrum is independent of the moduli and we display it in table 6.1. ($\mathbf{A}_a$ and $\mathbf{S}_a$ denote the antisymmetric resp. symmetric tensor representations with respect to $U(N_a)$, $SO(N_a)$ or $Sp(N_a)$.) Since $\bar{\sigma}\Omega$ exchanges a brane with its mirror brane, the Chan-Paton indices of strings ending on a stack of branes with non-vanishing gauge flux have no Ω projection and the gauge group is $U(N_a)$. If $\bar{\sigma}\Omega$ leaves branes invariant, i.e. the flux vanishes or is infinite, corresponding to pure D9- or D5-branes, the gauge factor is $SO(N_a)$ or $Sp(N_a)$, respectively.

The degeneracy of states stated in the third column of table 6.1 is essentially given by the intersection numbers of the D7-branes. Whenever it is formally negative, one has to pick the $(2,1)$ spinor of opposite chirality taking into account the opposite orientation of the branes at the intersection. As was pointed out earlier, a change of the orientation switches the RR charge in the tree channel translating into the opposite GSO projection in the loop channel. Therefore the other chirality survives the GSO projection in the R sector. If the multiplicity is zero, this does not mean that there are no massless open string

Spin	Representation (gauge group)	Multiplicity
$(1,2)$	$\mathbf{A}_a + \bar{\mathbf{A}}_a$	$2m_a^{(1)}m_a^{(2)}(n_a^{(1)}n_a^{(2)}+1)$ $= \frac{1}{2}(I_{a\,a'} + I_{a\,\mathrm{O7}})$
$(1,2)$	$\mathbf{S}_a + \bar{\mathbf{S}}_a$	$2m_a^{(1)}m_a^{(2)}(n_a^{(1)}n_a^{(2)}-1)$ $= \frac{1}{2}(I_{a\,a'} - I_{a\,\mathrm{O7}})$
$(1,2)$	$(\mathbf{N}_a, \overline{\mathbf{N}}_b) + (\overline{\mathbf{N}}_a, \mathbf{N}_b)$	$I_{a\,b}$
$(1,2)$	$(\mathbf{N}_a, \mathbf{N}_b) + (\overline{\mathbf{N}}_a, \overline{\mathbf{N}}_b)$	$I_{a\,b'}$

Table 6.1: Chiral 6D massless open string spectrum. The intersection form I is introduced in section 6.5.1.

states in this sector, it only means that the spectrum is not chiral. This happens precisely when two branes lie on top of each other in one of the two $T^2_{(j)}$ tori. Then the extra zero modes give rise to an extra spinor state of opposite chirality. The chiral spectrum shown in table 6.1 does indeed cancel the irreducible R^4 and F^4 anomalies.

We have also considered a $\mathbb{Z}_2$ orbifold background, together with non-vanishing magnetic flux, which changes the second condition in (6.35) to

$$\frac{R_2^{(1)}R_2^{(2)}}{R_1^{(1)}R_1^{(2)}}: \qquad \sum_{a=1}^{K} N_a\, m_a^{(1)} m_a^{(2)} = 16 \tag{6.37}$$

and leads to a projection $SO(N_a),\ Sp(N_a) \to U(N_a/2)$ on pure D9- and D5-branes but no further changes on D9-branes with non-vanishing flux. In this background it appears to be possible to construct also supersymmetric models [165].

6.4 Four dimensional models

The completely analogous computation as in six dimensions can be performed for the compactification of Type I strings on a 6-torus in the presence of additional gauge fields. Now we cancel the tadpoles by D9-branes with magnetic fluxes on all three 2-tori respectively, in the T-dual picture, by D6-branes at

Representation	Multiplicity
$(\mathbf{A}_a)_L$	$4m_a^{(1)}m_a^{(2)}m_a^{(3)}(n_a^{(1)}n_a^{(2)}n_a^{(3)}+1)$ $=\frac{1}{2}(I_{a'\,a}+I_{\mathrm{O6}\,a})$
$(\mathbf{S}_a)_L$	$4m_a^{(1)}m_a^{(2)}m_a^{(3)}(n_a^{(1)}n_a^{(2)}n_a^{(3)}-1)$ $=\frac{1}{2}(I_{a'\,a}-I_{\mathrm{O6}\,a})$
$(\overline{\mathbf{N}}_a,\mathbf{N}_b)_L$ $(\mathbf{N}_a,\mathbf{N}_b)_L$	$I_{a\,b}$ $I_{a'b}$

Table 6.2: Chiral 4D massless open string spectrum.

angles. One obtains four independent tadpole cancellation conditions

$$\begin{aligned}
\left(\prod_{j=1}^{3}\tau_2^{(j)}\right)^{-1} &= \frac{R_1^{(1)}R_1^{(2)}R_1^{(3)}}{R_2^{(1)}R_2^{(2)}R_2^{(3)}}: &\quad \sum_{a=1}^{K} N_a\, n_a^{(1)}n_a^{(2)}n_a^{(3)} &= 16 \\
\left(\tau_2^{(1)}\right)^{-1}\prod_{j=2}^{3}\tau_2^{(j)} &= \frac{R_1^{(1)}R_2^{(2)}R_2^{(3)}}{R_2^{(1)}R_1^{(2)}R_1^{(3)}}: &\quad \sum_{a=1}^{K} N_a\, n_a^{(1)}m_a^{(2)}m_a^{(3)} &= 0 \\
\tau_2^{(1)}\left(\tau_2^{(2)}\right)^{-1}\tau_2^{(3)} &= \frac{R_2^{(1)}R_1^{(2)}R_2^{(3)}}{R_1^{(1)}R_2^{(2)}R_1^{(3)}}: &\quad \sum_{a=1}^{K} N_a\, m_a^{(1)}n_a^{(2)}m_a^{(3)} &= 0 \\
\prod_{j=1}^{3}\tau_2^{(j)} &= \frac{R_2^{(1)}R_2^{(2)}R_1^{(3)}}{R_1^{(1)}R_1^{(2)}R_2^{(3)}}: &\quad \sum_{a=1}^{K} N_a\, m_a^{(1)}m_a^{(2)}n_a^{(3)} &= 0
\end{aligned} \tag{6.38}$$

(For convenience they are given in the picture with D6-branes at angles.) Again the gauge group contains a $U(N_a)$ factor for each stack of D9-branes with non-vanishing flux, an $SO(N_a)$ gauge factor for a stack with vanishing flux and an $Sp(N_a)$ factor for a stack of D5-branes. The general spectrum of chiral fermions with respect to the gauge group factors is presented in table 6.2. Whenever the intersection number in the second column is formally negative, one again has to take the conjugate representation. The spectrum in table 2 is free of non-abelian gauge anomalies.

In the next subsections we discuss some examples and point out some phenomenological issues for these models.

6.4.1 A 24 generation $SU(5)$ model

Having found a way to break supersymmetry, to reduce the rank of the gauge group and to produce chiral spectra in four space-time dimensions, it is tempting to search in a compact bottom-up approach for brane configurations producing massless spectra close to the Standard Model. The tachyons are not that dangerous from the effective field theory point of view, as they simply may serve as Higgs-bosons for spontaneous gauge symmetry breaking, anticipating a mechanism to generate a suitable potential keeping their vacuum expectation values finite. In [28] a three generation GUT model was presented, which we shall revisit

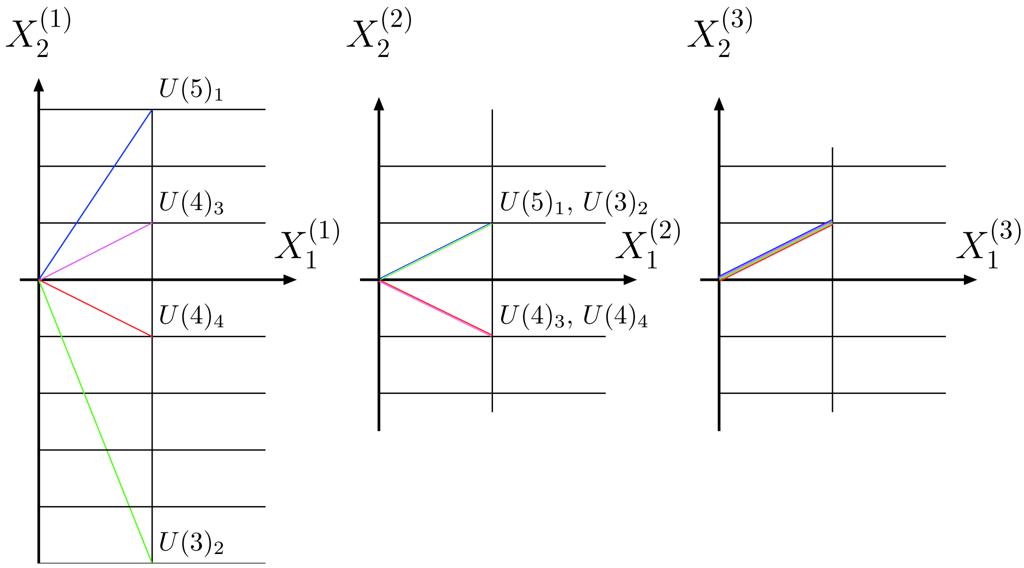

Figure 6.2: D6-brane configuration of the 24 generation model ($\bar{\sigma}$-pictures of D6 branes omitted).

in the following. The gauge group of the model is $G = U(5) \times U(3) \times U(4) \times U(4)$ with maximal rank, so that we have to choose all $n_a^{(j)} = 1$. The following choice of $m_a^{(j)}$ then satisfies all tadpole cancellation conditions (6.38):

$$m_a^{(j)} = \begin{pmatrix} 3 & -5 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & 1 & 1 & 1 \end{pmatrix} \tag{6.39}$$

This configuration of D6-branes is displayed in figure 6.2, where the mirror branes have been omitted. The chiral part of the fermionic massless spectrum is shown in table 6.3. No chiral fermions transform under both the $U(5) \times U(3)$ gauge group and the $U(4) \times U(4)$ gauge group, but there will of course also be non-chiral bifundamentals. If we think of the $SU(5)$ factor as a GUT gauge group, then this model has 24 generations[8]. We shall see in the following that it is actually impossible to get a model with three or any odd number of generations if we restrict all tori to admit a purely imaginary complex structure τ.

[8] In [28] this model was advocated as a three generation model. We can formally reproduce the model in [28] by dividing the matrix (6.39) by a factor of two. However, this is inconsistent as it would violate the condition that the $m_a^{(j)}$'s have to be integers. Thus, we conclude that in string theory only the choice (6.39) is correct and the model is actually a 24 generation model.

$U(5) \times U(3) \times U(4)^2$	Multipl.
$(\mathbf{10}, \mathbf{1}, \mathbf{1}, \mathbf{1})$	24
$(\mathbf{1}, \mathbf{3}, \mathbf{1}, \mathbf{1})$	40
$(\bar{\mathbf{5}}, \bar{\mathbf{3}}, \mathbf{1}, \mathbf{1})$	8
$(\mathbf{1}, \mathbf{1}, \bar{\mathbf{6}}, \mathbf{1})$	8
$(\mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{6})$	8

Table 6.3: Chiral left-handed fermions for 24 generation model

$SU(3) \times SU(2) \times U(1)^4$	Multipl.
$(\mathbf{3}, \mathbf{2})_{(1,1,0,0)}$	2
$(\mathbf{3}, \mathbf{2})_{(1,-1,0,0)}$	2
$(\bar{\mathbf{3}}, \mathbf{1})_{(-1,0,-1,0)}$	4
$(\bar{\mathbf{3}}, \mathbf{1})_{(-1,0,1,0)}$	4
$(\mathbf{1}, \mathbf{2})_{(0,1,0,1)}$	2
$(\mathbf{1}, \mathbf{2})_{(0,-1,0,1)}$	2
$(\mathbf{1}, \mathbf{1})_{(0,0,-1,-1)}$	4
$(\mathbf{1}, \mathbf{1})_{(0,0,1,-1)}$	4

Table 6.4: Chiral left-handed fermions for 4 generation model

6.4.2 A four generation model

The tadpole cancellation condition

$$\sum_{a=1}^{K} N_a \, n_a^{(1)} n_a^{(2)} n_a^{(3)} = 16 \tag{6.40}$$

tells us that we can reduce the rank of the gauge group right from the beginning by choosing some $n_a^{(j)} > 1$. Therefore, we can envision a model where we start with the gauge group $U(3) \times U(2) \times U(1)^r$ at the string scale. In order to have three quark generations in the $(\mathbf{3}, \mathbf{2})$ representation of $SU(3) \times U(2)$, we necessarily need $I_{12} = 3$ and $I_{12'} = 0$. It turns out that this is not possible if all three tori are of **A**-type as in this case $I_{ab} - I_{ab'}$ is always an even number. After publishing the paper, the **B**-torus as well as products of **A** and **B**-type tori were considered in [161]. It was shown that the obstruction of $I_{ab} - I_{ab'}$ =even can be evaded in these compactifications. There it was shown as well, that it is *not* possible to have $I_{12} = 3$ and $I_{12'} = 0$ [9] as well as RR-tadpole cancellation. However it is possible that *two* quark generations transform in the $(\mathbf{3}, \mathbf{2})$ and *one* generation in the $(\mathbf{3}, \bar{\mathbf{2}})$. This means that one quark generation has opposite $U(1)$-charge w.r.t. the $U(2)$ stack.[10]

Ibanez & al. used combinations of **A** and **B**-type tori in [163] to construct a class of (toroidal) models with chiral spectrum extremely close to the Standard Model.

The model we found in [2] closest to the 4 generation Standard Model is presented in the following. We choose the gauge group $U(3) \times U(2) \times U(1)^2$ and the following configuration of four stacks of D-branes:

$$n_a^{(j)} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 10 \end{pmatrix} \qquad m_a^{(j)} = \begin{pmatrix} 0 & 2 & 2 & 0 \\ 0 & 1 & -2 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix} \tag{6.41}$$

[9] This would imply that all three quark generations transform in the $(\mathbf{3}, \mathbf{2})$ of $U(3) \times U(2)$.

[10] The fundamental representation of $SU(2) \simeq SO(3)$ is (pseudo-)real. By the bar over the $\mathbf{2}$ we mean that the $U(2)$-representation has opposite $U(1)$ charge w.r.t. to the unbarred $\mathbf{2}$.

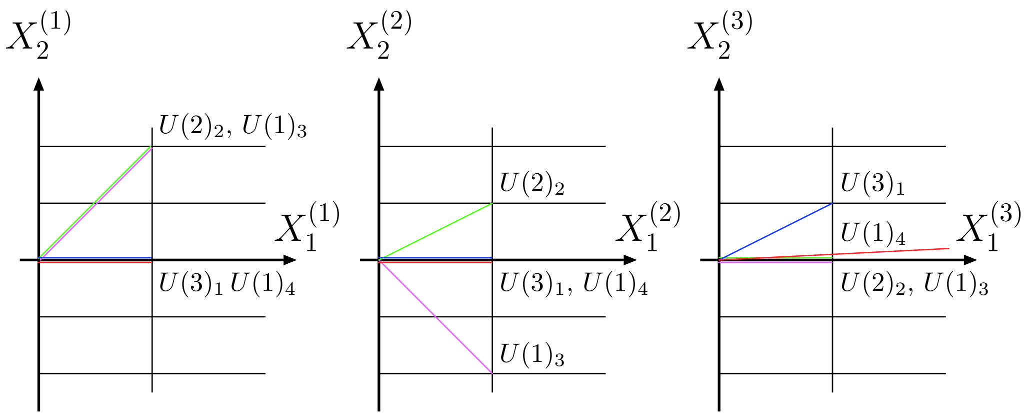

Figure 6.3: D6-brane configuration of the 4 generation $SU(3)\times SU(2)\times U(1)_Y\times U(1)^2$ model ($\bar{\sigma}$-pictures of D6 branes omitted).

The configuration has been illustrated in figure 6.3. The resulting chiral massless spectrum is shown in table 6.5. Computing the mixed $G^2-U(1)$ anomalies one realizes that one of the abelian gauge factors is anomalous, which needs to be cured by the Green-Schwarz mechanism. The other three anomaly-free abelian gauge groups include a suitable hypercharge $U(1)$

$$U(1)_Y = \frac{1}{3}U(1)_1 + U(1)_3 - U(1)_4 \tag{6.42}$$

so that the spectrum finally looks like the one in table 6.5. We found a semi-realistic, non-supersymmetric, four generation Standard Model like spectrum with two gauged flavor symmetries and right-handed neutrinos. In order to determine the Higgs sector, we would have to investigate the bosonic part of the spectrum. However, this is not universal but depends on the radii of the six-dimensional torus. We will not elaborate this further but instead discuss another important issue concerning the possible phenomenological relevance of these models.

$SU(3)\times SU(2)\times U(1)_Y\times U(1)^2$	Multiplicity
$(\mathbf{3},\mathbf{2})_{(\frac{1}{3},1,0)}$	2
$(\mathbf{3},\mathbf{2})_{(\frac{1}{3},-1,0)}$	2
$(\bar{\mathbf{3}},\mathbf{1})_{(-\frac{4}{3},0,-1)}$	4
$(\bar{\mathbf{3}},\mathbf{1})_{(\frac{2}{3},0,1)}$	4
$(\mathbf{1},\mathbf{2})_{(-1,1,0)}$	2
$(\mathbf{1},\mathbf{2})_{(-1,-1,0,)}$	2
$(\mathbf{1},\mathbf{1})_{(0,0,-1)}$	4
$(\mathbf{1},\mathbf{1})_{(2,0,1)}$	4

Table 6.5: Chiral left-handed fermions for 4 generation model including anomaly-free $U(1)$-charges

Since we break supersymmetry already at the string scale M_s, in order to solve the gauge hierarchy problem we must choose M_s in the TeV region. Let us employ the T-dual picture of D6-branes at angles again to analyze the situation in more detail. Using the relations

$$M_{\rm Pl}^2 \sim \frac{M_s^8 V_6}{g_s^2} \qquad\qquad \frac{1}{(g_{\rm YM}^{(a)})^2} \sim \frac{M_s^3 V_a}{g_s} \tag{6.43}$$

where V_a denotes the volume of some D6-brane in the internal directions

$$V_a = \prod_{j=1}^{3} V_a^{(j)} \tag{6.44}$$

and $g_{\rm YM}^{(a)}$ the gauge coupling on this brane. They imply

$$M_s \sim \alpha_{\rm YM}^{(a)} M_{\rm Pl} \frac{V_a}{\sqrt{V_6}} \tag{6.45}$$

Therefore, for the TeV scenario to work one needs

$$\frac{V_a}{\sqrt{V_6}} \ll 1 \tag{6.46}$$

for all D6-branes. However, chirality for the fermionic spectrum of an open string stretched between any two D6-branes implies that the two branes in question do not lie on top of each other on any of the three $T^2_{(j)}$ tori. In other words the two branes already span the entire torus and the condition (6.46) cannot be realized.

6.5 (In-) Stability of purely toroidal orientifolds

In this chapter we will make some comments about the stability of purely toroidal orientifolds. Stability usually demands the vanishing of the partition function, as the partition function in string theory is interpreted as a dilaton potential. To do consistent string perturbation theory, no tadpoles are allowed at any order of the string perturbation theory. (Otherwise one could hope that higher order contributions could "repair" a tadpole that originates from lower order terms.) Actually the dilaton does not couple to $\chi = 0$ amplitudes, however other string-excitations do. Supersymmetry guarantees the vanishing of the partition function[11]. Therefore imposing supersymmetry is a very general method to get stable models. Compactifications on flat tori, which are the starting point of our construction, have a supersymmetric closed string sector. The three unoriented $\chi = 0$ diagrams (Klein-bottle, Cylinder and Möbius strip) are divergent only due to closed string tadpoles. However also in the open string sector we can often isolate a single excitation that is responsible for an

[11]In the light-cone quantized string on the pp-wave the partition function does not vanish, but this is believed to be connected to the fact that the light-cone gauge does not cover all string states (i.e. $p^+ = 0$). (Private communication with Matthias Gaberdiel.)

instability: the tachyon. Due to the tachyon relation $M^2_{\text{tach}} < 0$ it has been suggested that the tachyon can condense like a Higgs field ϕ whose potential $V(\phi) \propto -\mu^2|\phi|^2 + \lambda^2|\phi|^4$ if expanded around $\phi = 0$ contains a negative mass^2-term as well. Tachyon condensation in terms of *string field theory* has been investigated in a variety of papers. We only mention two of the first [166, 167], a complete list would require more than hundred entries. However we will mention the conditions for absence of tachyons, which where published after the paper this chapter is based on, was finished. First however we show that *chiral* supersymmetric $\Omega\bar{\sigma}$ orientifolds of the torus are impossible. This restriction does not apply for all toroidal orbifolds.[12] The first example of a four dimensional, chiral supersymmetric $\Omega\bar{\sigma}$-orientifold was the $\mathbb{Z}_2 \times \mathbb{Z}_2$ orientifold investigated by Cvetic & al. which in addition has the nice feature to admit quasi-realistic models. The second example is the $\mathbb{Z}_4$ $\Omega\bar{\sigma}$-orientifold on a six torus [57] which admits chiral supersymmetric models as well [43, 3]. Some of them are even phenomenological interesting.[13] We will postpone this discussion to the next chapter.

6.5.1 Supersymmetric brane configurations and special Lagrangian submanifolds (sLags)

The mathematical foundations of calibrations and sLags are described for example in the classic publication [169] and a newer article by Joyce [170]. Here we will state some important definitions and consequences. Without going into the details, we state the common fact that D-branes that are wrapped around so called *special Lagrangian submanifolds* (sLags) preserve some amount of the bulk (or closed string) supersymmetry [102], if no NS F field is switched on and if no NSNS B-field component is along the D-brane. There are many ways to introduce the notion of a sLag. One is to start by defining what a *calibration* is. Roughly speaking a calibration is a *closed* k-form ϕ on a manifold $\mathcal{M}$ (with volume form vol) such that given any oriented k plane V in the tangent bundle $T\mathcal{M}$ the induced k-volume $\text{vol}_k|_V$ form is always bigger or equal than the restriction of ϕ:

$$\phi|_V \leq \text{vol}_k|_V \tag{6.47}$$

In this sense a ϕ-calibrated submanifold $\mathcal{N} \subset \mathcal{M}$ is a submanifold of dimension k s.th. for all oriented tangent planes $T_x\mathcal{N}$ of $\mathcal{N}$ the restriction of ϕ equals the restriction of the volume form:

$$\phi|_{T_x\mathcal{N}} = \text{vol}_k|_{T_x\mathcal{N}} \qquad \forall x \in \mathcal{N} \tag{6.48}$$

The striking result (of a theorem) is that a calibrated submanifold is always volume-minimizing in its homology class.

CYn-spaces $\mathcal{M}$ (cf. 2.3.1, p. 45) consist of a quadruple[14] $(\mathcal{M}, J, g, \Omega)$. $\mathcal{M}$

[12] However the $\mathbb{Z}_3$ orientifold investigated in [59] is always non-supersymmetric for chiral models.

[13] Meanwhile similar features have been found in the $\Omega\bar{\sigma}$-orientifold of the $\mathbb{Z}_4 \times \mathbb{Z}_2$-orbifold [168].

[14] This follows either from the definition of the CY space, or if the CY space is introduced differently, by further theorems.

is the complex compact manifold (of complex dimension n) itself, J the corresponding complex structure. g is the Kähler metric with the *Levi-Civita connection* leading to $SU(n)$ holonomy. (The Kähler form ω is is obtained from J and g). Ω is a non-zero covariantly constant $(n,0)$-form s.th.:

$$\frac{\omega^m}{m!} = (-1)^{m(m-1)/2}\Big(\frac{i}{2}\Big)^m \Omega\wedge\bar{\Omega} \tag{6.49}$$

By the above definition of Ω the real part $\mathfrak{Re}(\Omega)$ is automatically a calibration on $\mathcal{M}$.[15] A $\mathfrak{Re}(\Omega)$-calibrated submanifold of a CYn-fold is called a "special Lagrangian submanifold" (sLag). SLags have real dimension n.

A theorem states that given a CYn-fold $\mathcal{M}$ (defined as above) and a real n-dimensional submanifold $\mathcal{N}$, then $\mathcal{N}$ admits an orientation making it into a sLag iff both $\omega|_{\mathcal{N}} = 0$ and $\mathfrak{Im}(\Omega|_{\mathcal{N}}) = 0$ are fulfilled.[16]

In $\mathbb{C}^2$ the sLags are given by holomorphic curves.

The case of $\bar{\sigma}\Omega$ orientifolds with sLags has been studied in great generality in [43]. We will now take advantage of some of the facts inherited by sLags. Applying these properties to the four dimensional case of compactification on $T^2\times T^2$, Ω is given by $dz_1\wedge dz_2$. The D6-branes projected on this four-torus are real two-dimensional. We have restricted ourselves to flat branes, each a product of two one-cycles, a one-cycle on each two-torus $T^2_{(j)}$. The sLag condition then gives rise to the condition that the sum of the two oriented angles vanishes:

$$\phi^{(1)} + \phi^{(2)} = 0 \tag{6.50}$$

The O7-plane, i.e. the fixed locus of $\bar{\sigma}$ has $\phi_1 = \phi_2 = 0$.[17] In other words: the O7-plane is parallel to the $X_1^{(i)}$-axis. Canceling the RR-tadpole means that the complete holonomy-class of the cycle associated with the D-branes cancels exactly the holonomy of the O7-plane:[18]

$$\sum_a N_a(\pi_a + \pi_{a'}) = 8\pi_{\mathrm{O7}} \tag{6.51}$$

The π_a are elements of the homology generated by the basis (cf. figure 6.1):[19]

$$\overline{\langle p_1 \equiv \pi_1\otimes\pi_3, p_2 \equiv \pi_2\otimes\pi_4, p_3 \equiv \pi_2\otimes\pi_3, p_4 \equiv \pi_1\otimes\pi_4\rangle} \oplus \overline{\langle \pi_1\otimes\pi_2, \pi_3\otimes\pi_4\rangle} = H_2\big(T^4\big) \tag{6.52}$$

[15] $\mathfrak{Re}(\exp(i\theta)\Omega)$ is another calibration form. In what follows, we will consider only calibrations w.r.t. $\exp(i\theta) = 1$.

[16] If only the condition $\omega|_{\mathcal{N}} = 0$ is obeyed, $\mathcal{N}$ is called a "Lagrangian submanifold". The supplement "special" means the additional property $\mathfrak{Im}(\Omega|_{\mathcal{N}}) = 0$. Langrangian submanifolds are defined more generally in symplectic geometry, where ω denotes the symplectic form.

[17] The fact that $O(9-d)$-planes can be interpreted as fixed loci of $\bar{\sigma}$ is explained in [43].

[18] The factor of 8 in front of π_{O7} is determined by the tadpole cancellation conditions (6.35). $\pi_{a'}$ denotes the $\bar{\sigma}$-image of π_a.

[19] We have split the basis. Element of the second summand do not appear in our models.

In the p-basis the cycle wrapped by the D7-brane a is expressed by:[20]

$$\pi_a = \begin{pmatrix} n_a^{(1)} n_a^{(2)} \\ m_a^{(1)} m_a^{(2)} \\ m_a^{(1)} n_a^{(2)} \\ n_a^{(1)} m_a^{(2)} \end{pmatrix} \qquad I = \left(\begin{array}{cc|cc} 0 & 1 & \multicolumn{2}{c}{\multirow{2}{*}{0}} \\ 1 & 0 & & \\ \hline \multicolumn{2}{c|}{\multirow{2}{*}{0}} & 0 & 1 \\ & & 1 & 0 \end{array}\right) \tag{6.53}$$

In the above basis the homology class of the O7-plane (i.e. the $\bar{\sigma}$-invariant locus) is given by:

$$\pi_{\mathrm{O7}} = 4\, p_1 = (4, 0, 0, 0)^T \tag{6.54}$$

It is now clear that the configuration with the smallest volume that lies in the same Homology class as the O7 plane, is the one in which all D-branes are parallel to the $X_1^{(i)}$ axis. Any flat tilted brane with identical π_{O7}-component and non-zero angle wrt. the $X_1^{(i)}$-axis will have larger volume. Such a brane configuration can not be a sLag since sLags are volume minimizing in their homology class. Thus, in the absence of NSNS B-fields with components parallel to the branes and in the absence of an NS F-flux, the only supersymmetry preserving D7-brane configurations that cancel the RR-tadpole, are the ones where all branes are parallel to the O7-plane.[21] These configurations are however the ones without chiral fermions.

The same arguments go through for the six-torus (i.e. the four-dimensional models). In contrast to the four compact dimensions, the six-dimensional torus forces B-field components along the D6-brane to vanish, as well as $F = 0$ for the NS $U(1)$-fields. This was shown in [103]. Therefore we conclude that pure toroidal compactifications can not reconcile both supersymmetry and chirality w.r.t. the gauge group.

In the next section, we summarize what has been found out on the existence and non-existence of (open-) string tachyons in toroidal compactifications.

6.5.2 Tachyons in toroidal orientifolds

As we already noted, tachyons can only appear in the open string sector, as the closed string sector is supersymmetric. In our original publication we concluded that in six-dimensional models open string tachyons generically appear due to the fact that two D7-branes that are rotated by an angle (ϕ_1, ϕ_2) always imply negative ground state energy, except for the supersymmetric case $\phi_1 = -\phi_2$. Thus in $\Omega\bar{\sigma}$-orientifolds on T^4 we always encounter one or several open string tachyons, iff supersymmetry is broken. However for open strings

[20] I denotes the intersection matrix. It can be used to calculate the intersection number as well (cf. table 6.1).

[21] However the brane might be deformed, s.th. it is no longer flat: a theorem by McLean states that the dimension of the moduli space of a sLag $\mathcal{N}$ equals the first Betti number $b^1(\mathcal{N})$ (cf. [171, 170]). In the case of the flat tow-torus however, $b^1(\mathcal{N})$ just equals the number of independent translations of the D-brane in its normal direction, which is two. Therefore we conclude that a deformation that transforms a flat D-brane to a non-flat one, will spoil the sLag condition and as a consequence will break supersymmetry. Similar to the CY-case where the geometric Kähler cone gets complexified by the NSNS B-field, we can add Wilson lines,

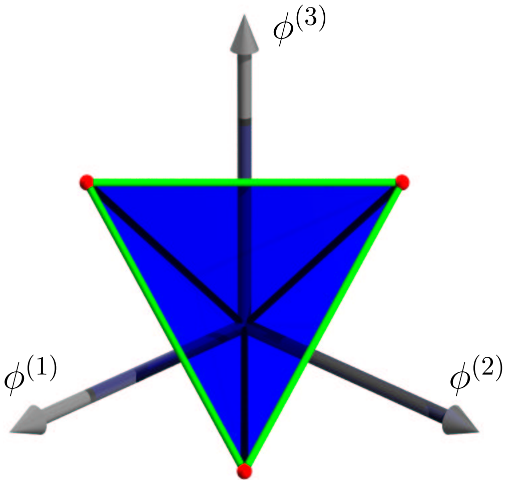

Figure 6.4: Tachyon region: Tachyons develop outside the tetrahedron. The vertices (red) preserve $\mathcal{N} = 4$, and the edges (green) $\mathcal{N} = 2$ supersymmetry. The faces (transparent blue) only preserve $\mathcal{N} = 1$. While inside the tetrahedron ($\mathcal{N} = 0$) no open string tachyons are present, the brane configuration generically destabilizes due to closed string tadpoles in this case. The four vertices sit at $(\phi^{(1)}, \phi^{(2)}, \phi^{(3)})$=$(0, 0, 0)$, $(\pi, \pi, 0)$, $(0, \pi, \pi)$, $(\pi, 0, \pi)$.

stretching between two D6-branes with angles ($\phi_1 \neq 0, \phi_2 \neq 0, \phi_3 \neq 0$) tachyon free regions exist in the parameter space ([162, 163, 164]). We represent the distinct cases in figure 6.4 (cf. [163]). The tachyon free region is inside the tetrahedron. The vertices, edges and faces are not only tachyon free, but carry also different amounts of supersymmetry. The vertices preserve $\mathcal{N} = 4$, the edges $\mathcal{N} = 2$ and the faces $\mathcal{N} = 1$. However different faces carry different kinds of $\mathcal{N} = 1$ supersymmetry which is determined by the signs of the supersymmetry condition: $\phi_1 \pm \phi_2 \pm' \phi_3 = 0$. Similarly, two adjacent faces carry only one common supersymmetry. The vertices are however equivalent, as the situation is invariant under shifting the tetrahedron by lattice-vectors of the type:

$$\Gamma = \{l \cdot (\pi, \pi, 0) + m \cdot (0, \pi, \pi) + n \cdot (\pi, 0, \pi) \, | l, n, m \in \mathbb{Z}\} \tag{6.55}$$

In other words, branes rotated in exactly two complex planes by angles π preserve the same supersymmetry. (A rotation by π in a single plane would transform the brane to an antibrane.) Actually the tetrahedron in figure 6.4 should be repeated each lattice vector, which we did not do for clarity of the picture. Outside the tetrahedron tachyons are present. It turned out to be possible to build tachyon-free toroidal models with a chiral spectrum extremely close to the Standard Model [163]. Even thought tachyons might be absent, a non-supersymmetric orientifold-model is unstable if there are uncanceled NSNS-tadpoles. If the closed string moduli corresponding to the tadpoles are related to the angles of the branes, the model can be driven to point where open string tachyons appear. In the toroidal models complex-structure moduli influence the angles of the branes, and these are exactly the ones which develop NSNS tadpoles (besides the dilaton).

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i.e. flat gauge connections, without breaking supersymmetry. The number of independent Wilson lines equals $\dim(\pi_1)$ which in turn again equals $b^1(\mathcal{N})$ (cf. [31]).

Concluding remarks

In this paper we have investigated Type I string compactifications on non-commutative tori, which are due to constant magnetic fields along the world volumes of the D9-branes being wrapped around the internal space. The number of chiral fermions (arising from strings "stretched" between two branes with different NS F-field-strength on each T^2) can be interpreted as a Landau degeneracy. It is given by the *Atiyah-Singer index theorem* for twisted spin-complexes.

In the T-dual picture the magnetized D9-branes become lower-dimensional branes, while the Ω-parity projection now includes a complex conjugation on all T^2's: $\Omega \xrightarrow{\text{T-duality}} \bar{\sigma}\Omega$. In this picture, the chiral fermions are located at the brane intersection-points. The analog of the Landau degeneracy is the topological intersection number of two D-branes.

In the given setting, we found a four-dimensional model with Standard Model gauge group $SU(3) \times SU(2) \times U(1)_Y$ (times some abelian flavor gauge groups) with four generations of Standard Model fermions and also a four-dimensional 24 generation $SU(5)$ GUT-like model. However it turned out, that the Ansatz $T^6 = \prod_j T^2_{(j)\,\mathbf{A}}$ involving only $\mathbf{A}$-type tori leads always to an even number of left handed quark generation, as the difference of the intersection number $I_{ab} - I_{ab'}$ is always even. As shown in a subsequent publication [161], this problem can be evaded by considering more general T^6-tori involving $\mathbf{B}$-type two-tori, as well.

We found that in chiral toroidal orientifolds, supersymmetry is always broken. This could lead to open string tachyons. However many models have been found after our publication, in which tachyons are absent (cf. [163]). Nevertheless, closed string tadpoles tend to destabilize the model and drive it to a singular limit of the T^6. Dilaton tadpoles can in general push a non-supersymmetric model to either a strongly-coupled or free regime. Investigations if a non-supersymmetric string model could be stabilized by the *Fischler-Susskind mechanism* [54, 55] have been undertaken as well (cf. [172, 173]).

The stability problems would be absent, if one can construct supersymmetric models. That it is indeed possible to reconcile both supersymmetry and chirality has been shown by Cvetic & al. [174] who considered a (left-right symmetric) $T^6/(\mathbb{Z}_2 \times \mathbb{Z}_2)$ Type IIA orientifold. This orientifold has phenomenologically appealing solutions. The second example of a Type IIA orientifold involving branes at angles and admitting chiral supersymmetric solutions was published by us some time later (cf. [3]). We found a solution with a gauge-group that can be broken down to the SM gauge group in a supersymmetry preserving way by giving VEVs to fields in the low energy effective action. The matter content of the resulting model is extremely close to the MSSM. We will review the $T^6/\mathbb{Z}_4$ $\bar{\sigma}\Omega$-orientifold in the next chapter.

Chapter 7

The $\bar{\sigma}\Omega$-Orientifold on $(T^2 \times T^2 \times T^2)/\mathbb{Z}_4$

Intersecting brane world models have been the subject of elaborate string model building for several years [2,165,124,175,162,176,161,163,177,164,59,178,174, 179,180,181,182,183,184,185,186,187,188,189,190,191,192,193,194,43,195,196, 197,198,199,200,201,202]. The main new ingredient in these models is that they contain intersecting D-branes and open strings in a consistent manner providing simple mechanisms to generate chiral fermions and to break supersymmetry [28,27]. Most attempts for constructing realistic models were dealing with non-supersymmetric configurations of D-branes, mainly because non-trivial, chiral supersymmetric intersecting brane world models are not easy to find. It is known for instance that flat factorizing D-branes on the six-dimensional torus as well as on the $T^6/\mathbb{Z}_3$ orbifold can never give rise to supersymmetric models except for the trivial non-chiral configuration where all D6-branes are located on top of the orientifold plane [59]. Supersymmetric models clearly have some advantages over the non supersymmetric ones. From the stringy point of view such models are stable, as not only the Ramond-Ramond (R-R) tadpoles cancel but also the Neveu-Schwarz-Neveu-Schwarz (NS-NS) tadpoles. From the phenomenological point of view, since the gauge hierarchy problem is solved by supersymmetry, one can work in the conventional scenarios with a large string scale close to the Planck scale or in an intermediate regime [203]. For an overview on other Type I constructions see [18].

The only semi-realistic supersymmetric models that have been found so far are defined in the $T^6/\mathbb{Z}_2 \times \mathbb{Z}_2$ orientifold background and were studied in a series of papers [178,174,192,193,200].[1] Besides their phenomenological impact, Type IIA supersymmetric intersecting brane worlds with orientifold six-planes and D6-branes are also interesting from the stringy point of view, as they are expected to lift to M-theory on singular G_2 manifolds [204].

The aim of this chapter (and the underlying publication [3]) is to pursue the study of intersecting brane worlds on orientifolds with a particular emphasis on the systematic construction of semi-realistic supersymmetric configurations.

[1] Meanwhile the $\mathbb{Z}_4 \times \mathbb{Z}_2$ orientifold with $\bar{\sigma}\Omega$-projection has been studied. Interesting non-chiral supersymmetric solutions have been found as well (cf. [168]).

Note, that without the orientifold projection supersymmetric intersecting brane configurations do not exist, as the overall tension always would be positive. Interestingly, from the technical point of view, the $\mathbb{Z}_4$ orbifold involves some new insights, as not all 3-cycles are inherited from the torus. In fact, a couple of 3-cycles arise in the $\mathbb{Z}_2$ twisted sector implying that this model contains so-called fractional D6-branes, which have been absent in the $\mathbb{Z}_2 \times \mathbb{Z}_2$ and $\mathbb{Z}_3$ orbifolds. To treat these exceptional cycles accordingly, we will make extensive use of the formalism developed in [43].

It will turn out that supersymmetric models in general can be constructed in a straightforward way. But as in other model building approaches, finding semi-realistic three generation models turns out to be quite difficult. Fortunately, we will finally succeed in constructing a globally supersymmetric three generation Pati-Salam model with gauge group $SU(4) \times SU(2)_L \times SU(2)_R$ and the Standard Model matter in addition to some exotic matter in the symmetric and antisymmetric representation of the two $SU(2)$ gauge groups. In this chapter, we will mainly focus on the new and interesting string model building aspects and leave a detailed investigation of the phenomenological implications of the discussed models for future work.

This chapter is organized as follows. In section 7.1 we review some of the material presented in [43] about the general structure of intersecting brane worlds on Calabi-Yau manifolds. We will review those formulæ which will be extensively used in the rest of the paper. In section 7.2 we start to investigate the $\mathcal{X} = T^6/\mathbb{Z}_4$ orbifold and in particular derive an integral basis for the homology group $H_3(\mathcal{X}, \mathbb{Z})$, for which the intersection form involves the Cartan-matrix of the Lie-algebra E_8. The main ingredient in the construction of such an integral basis will be the physical motivated introduction of fractional D-branes which also wrap around exceptional (twisted) 3-cycles in $\mathcal{X}$. In section 7.3 we construct the orientifold models of Type IIA on the orbifold $\mathcal{X}$ and discuss the orientifold planes, the action of the orientifold projection on the homology and the additional conditions arising for supersymmetric configurations. In section 7.4 we construct as a first example a globally supersymmetric four generation Pati-Salam model. Finally, in section 7.5 we elaborate on a supersymmetric model with initial gauge symmetry $U(4) \times U(2)^3 \times U(2)^3$ and argue that by brane recombination it becomes a supersymmetric three generation Pati-Salam model. By using conformal field theory methods, for this model we determine the chiral and also the massless non-chiral spectrum, which turns out to provide Higgs fields in just the right representations in order to break the model down to the Standard Model. At the end of the paper we describe both the GUT breaking and the electroweak breaking via brane recombination processes. We also make a prediction for the Weinberg angle at the string scale.

7.1 Intersecting Brane Worlds on Calabi-Yau spaces

Before we present our new model, we would like to briefly summarize some of the results presented in [43] about Type IIA orientifolds on smooth Calabi-Yau spaces. If the manifold admits an anti-holomorphic involution $\bar{\sigma}$, the combination $\Omega\bar{\sigma}$ is indeed a symmetry of the Type IIA model. Taking the quotient with respect to this symmetry introduces an orientifold six-plane into the background, which wraps a special Lagrangian 3-cycle of the Calabi-Yau.[2] In order to cancel the induced RR-charge, one introduces stacks of N_a D6-branes which are wrapped on 3-cycles π_a. Since under the action of $\bar{\sigma}$ such a 3-cycle, π_a, is in general mapped to a different 3-cycle, π'_a, one has to wrap the same number of D6-branes on the latter cycle, too. The equation of motion for the RR 7-form implies the RR-tadpole cancellation condition,

$$\sum_a N_a\,(\pi_a + \pi'_a) - 4\,\pi_{O6} = 0. \tag{7.1}$$

If it is possible to wrap a connected smooth D-brane on such a homology class, the stack of D6-branes supports a $U(N_a)$ gauge factor. Note, that it is not a trivial question if in a given homology class such a connected smooth manifold does exist. However, as we will see in section 7.5 for special cases, there are physical arguments ensuring that such smooth D-branes exist.

The Born-Infeld action provides an expression for the open string tree-level scalar potential which by differentiation leads to an equation for the NS-NS tadpoles

$$V = T_6\,\frac{e^{-\phi_4}}{M_s^3\sqrt{\mathrm{Vol}(\mathcal{X})}}\Big(\sum_a N_a\left(\mathrm{Vol}(\mathrm{D6}_a) + \mathrm{Vol}(\mathrm{D6}'_a)\right) - 4\,\mathrm{Vol}(\mathrm{O6})\Big) \tag{7.2}$$

with the four-dimensional dilaton given by $e^{-\phi_4} = M_s^3\sqrt{\mathrm{Vol}(\mathcal{X})}e^{-\phi_{10}}$ and T_6 denoting the tension of the D6-branes. By $\mathrm{Vol}(\mathrm{D6}_a)$ we mean the three dimensional internal volume of the D6-branes. Generically, this scalar potential is non-vanishing reflecting the fact that intersecting branes do break supersymmetry. If the cycles (more precisely: the corresponding submanifolds) are special Lagrangian (sLag) but calibrated with respect to 3-forms $\mathfrak{Re}(e^{i\theta}\Omega_{3,0})$ with different constant phase factors $\exp(i\theta)$, the expression gets simplified to[3]

$$V = T_6\,e^{-\phi_4}\left(\sum_a N_a\left|\int_{\pi_a}\widehat{\Omega}_{3,0}\right| + \sum_a N_a\left|\int_{\pi'_a}\widehat{\Omega}_{3,0}\right| - 4\left|\int_{\pi_{O6}}\widehat{\Omega}_{3,0}\right|\right) \tag{7.3}$$

In this case, all D6-branes preserve some supersymmetry but not all of them the same. Models of this type have been discussed in [184, 186]. In the case of a completely supersymmetric model, all 3-cycles are calibrated with respect to the same 3-form as the O6-plane implying that the disc level scalar potential vanishes due to the RR-tadpole condition (7.1).

In [43] it was argued and confirmed by many examples that the chiral massless spectrum charged under the $U(N_1)\times\ldots\times U(N_k)$ gauge group of a configuration of k intersecting stacks of D6-branes can be computed from the topological intersection numbers as shown in table 7.1. Since in six dimensions the in-

[2] By "special Lagrangian 3-cycle" we mean the real three-dimensional special Lagrangian submanifold, that lies in the same homology class as the 3-cycle.

[3] $\widehat{\Omega}_{3,0} = e^{i\theta}\Omega_{3,0}$

Representation	Multiplicity
$[\mathbf{A_a}]_L$	$\frac{1}{2}\left(\pi'_a \circ \pi_a + \pi_{\mathrm{O6}} \circ \pi_a\right)$
$[\mathbf{S_a}]_L$	$\frac{1}{2}\left(\pi'_a \circ \pi_a - \pi_{\mathrm{O6}} \circ \pi_a\right)$
$[(\mathbf{\bar{N}_a}, \mathbf{N_b})]_L$	$\pi_a \circ \pi_b$
$[(\mathbf{N_a}, \mathbf{N_b})]_L$	$\pi'_a \circ \pi_b$

Table 7.1: Chiral spectrum in $d = 4$

tersection number between two 3-cycles is anti-symmetric, the self intersection numbers do vanish implying the absence of chiral fermions in the adjoint representation. Negative intersection numbers correspond to chiral fermions in the conjugate representations. Note, that if we want to apply these formulæ to orientifolds on singular toroidal quotient spaces, the intersection numbers have to be computed in the orbifold space and not simply in the ambient toroidal space. After these preliminaries, we will discuss the $\mathbb{Z}_4$ orientifold in the following sections.

7.2 3-cycles in the $\mathbb{Z}_4$ orbifold

We consider Type IIA string theory compactified on the orbifold background $T^6/\mathbb{Z}_4$, where the action of the $\mathbb{Z}_4$ symmetry, Θ, on the internal three complex coordinates reads

$$z_1 \to e^{\frac{\pi i}{2}} z_1, \quad z_2 \to e^{\frac{\pi i}{2}} z_2, \quad z_3 \to e^{-\pi i} z_3 \tag{7.4}$$

with $z_1 = x_1 + ix_2$, $z_2 = x_3 + ix_4$ and $z_3 = x_5 + ix_6$. This action preserves $\mathcal{N} = 2$ supersymmetry in four dimensions so that the orbifold describes a singular limit of a Calabi-Yau threefold. The Hodge numbers of this threefold are given by $h_{21} = 7$ and $h_{11} = 31$, where 1 complex- and 5 Kähler-moduli arise in the untwisted sector. The Θ and Θ^3 twisted sectors contain 16 $\mathbb{Z}_4$ fixed points giving rise to 16 additional Kähler moduli. In the Θ^2 twisted sector, there are 16 $\mathbb{Z}_2$ fixed points from which 4 are also $\mathbb{Z}_4$ fixed points. The latter ones contain 4 Kähler moduli whereas the remaining twelve $\mathbb{Z}_2$ fixed points are organized in pairs under the $\mathbb{Z}_4$ action giving rise to 6 complex- and 6 Kähler-moduli. The fact that the $\mathbb{Z}_2$ twisted sector contributes $h_{21}^{\mathrm{tw}} = 6$ elements to the number of complex structure deformations and therefore contains what might be called twisted 3-cycles, is the salient new feature of this $\mathbb{Z}_4$ orbifold model as compared to the intersecting brane world models studied so far.

Given this supersymmetric closed string background, we take the quotient by the orientifold projection $\Omega\bar{\sigma}$, where $\bar{\sigma}$ is an anti-holomorphic involution $z_i \to e^{i\phi_i}\bar{z}_i$ of the manifold. Note, that this orientifold model is not T-dual to the $\mathbb{Z}_4$ Type IIB orientifold model studied first in [40]. In the latter model there did not exist any supersymmetric brane configurations canceling all tadpoles induced by the orientifold planes. In fact, as was pointed out in [1] our model is

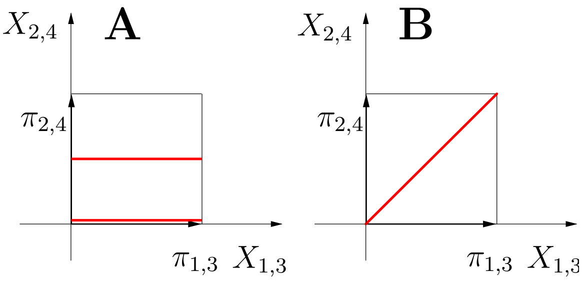


Figure 7.1: Anti-holomorphic involutions. The O6-planes, i.e. the fixed loci under $\bar{\sigma}$ are painted in red.

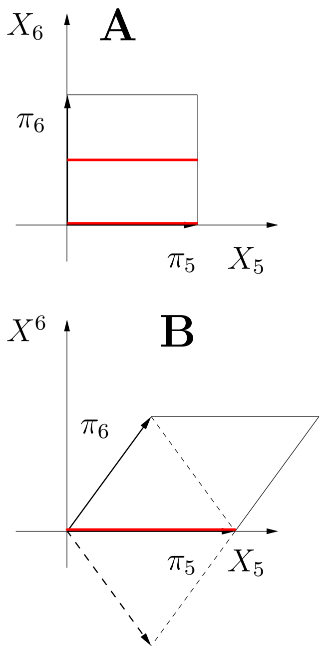


Figure 7.2: Orientations of the third T^2

T-dual to a Type IIB orientifold on an asymmetric $\mathbb{Z}_4$ orbifold space. Slightly different $\mathbb{Z}_4$ Type IIB orientifold models were studied in [153, 205].

Our orientifold projection breaks supersymmetry in the bulk to $\mathcal{N} = 1$ and introduces an orientifold O6-plane located at the fixed point locus of the anti-holomorphic involution. The question arises if one can introduce D6-branes, generically not aligned to the orientifold plane, in order to cancel the tadpoles induced by the presence of the O6 plane. The simplest such model where the D6-branes lie on top of the orientifold plane has been investigated in [57].

7.2.1 Crystallographic actions

Before dividing Type IIA string theory by the discrete symmetries $\mathbb{Z}_4$ and $\Omega\bar{\sigma}$, we have to ensure that the torus T^6 does indeed allow crystallographic actions of these symmetries. For simplicity, we assume that T^6 factorizes as $T^6 = T^2 \times T^2 \times T^2$. On the first two T^2s the $\mathbb{Z}_4$ symmetry enforces a rectangular torus with complex structure $U = 1$. On each torus two different anti-holomorphic involutions

$$\begin{aligned} \mathbf{A} &: z_i \rightarrow \bar{z}_i \\ \mathbf{B} &: z_i \rightarrow e^{i\frac{\pi}{2}}\,\bar{z}_i \end{aligned} \tag{7.5}$$

do exist. These two cases are shown in figure 7.1, where we have indicated the fixed point set of the orientifold projection $\Omega\bar{\sigma}$.[4]

[4]The same distinction between the involutions **A** and **B** occurred for the first time in the papers [56, 117, 57, 206].

Since on the third torus the $\mathbb{Z}_4$ acts like a reflection, its complex structure is unconstrained. But again there exist two different kinds of involutions, which equivalently correspond to the two possible choices of the orientation of the torus as shown in figure 7.2. For the **A**-torus its complex structure is given by $U = iU_2$ with U_2 unconstrained and for the **B**-torus the complex structure is given by $U = \frac{1}{2} + iU_2$. Therefore, by combining all possible choices of complex conjugations we get eight possible orientifold models. However, taking into account that the orientifold model on the $\mathbb{Z}_4$ orbifold does not only contain the orientifold planes related to $\Omega\bar{\sigma}$ but also the orientifold planes related to $\Omega\bar{\sigma}\Theta$, $\Omega\bar{\sigma}\Theta^2$ and $\Omega\bar{\sigma}\Theta^3$, only four models $\{$**AAA**,**ABA**,**AAB**,**ABB**$\}$ are actually different.

7.2.2 A non-integral basis of 3-cycles

In order to utilize the formulæ from section 7.1, we have to find the independent 3-cycles on the $\mathbb{Z}_4$ orbifold space. Since we already know that the third Betti number, $b_3 = 2 + 2h_{21}$, is equal to sixteen, we expect to find precisely this number of independent 3-cycles.

One set of 3-cycles we get for free as they descend from the ambient space. Consider the three-cycles inherited from the torus T^6. We call the two fundamental cycles on the torus T_I^2 $(I = 1, 2, 3)$ π_{2I-1} and π_{2I} and moreover we define the toroidal 3-cycles

$$\pi_{ijk} \equiv \pi_i \otimes \pi_j \otimes \pi_k. \tag{7.6}$$

Taking orbits under the $\mathbb{Z}_4$ action, one can deduce the following four $\mathbb{Z}_4$ invariant 3-cycles

$$\begin{aligned} \rho_1 &\equiv 2(\pi_{135} - \pi_{245}), \qquad & \bar{\rho}_1 &\equiv 2(\pi_{136} - \pi_{246}) \\ \rho_2 &\equiv 2(\pi_{145} + \pi_{235}), \qquad & \bar{\rho}_2 &\equiv 2(\pi_{146} + \pi_{236}) \end{aligned} \tag{7.7}$$

The factor of two in (7.7) is due to the fact that Θ^2 acts trivially on the toroidal 3-cycles. In order to compute the intersection form, we make use of the following fact: if the 3-cycles π_a^t on the torus are arranged in orbits of length N under some $\mathbb{Z}_N$ orbifold group, i.e.

$$\pi_a \equiv \sum_{i=0}^{N-1} \Theta^i \pi_a^t \tag{7.8}$$

the intersection number between two such 3-cycles on the orbifold space is given by

$$\pi_a \circ \pi_b = \frac{1}{N} \left(\sum_{i=0}^{N-1} \Theta^i \pi_a^t \right) \circ \left(\sum_{j=0}^{N-1} \Theta^j \pi_b^t \right) \tag{7.9}$$

Therefore, the intersection form for the four 3-cycles (7.7) reads

$$I_\rho = \bigoplus_{i=1}^{2} \begin{pmatrix} 0 & -2 \\ 2 & 0 \end{pmatrix} \tag{7.10}$$

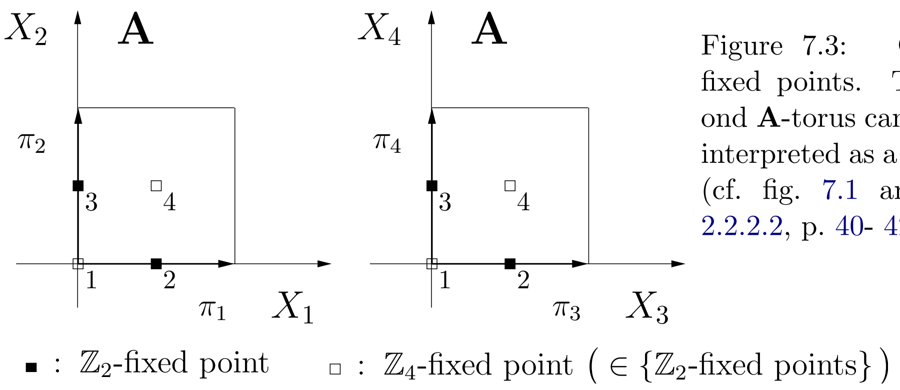


Figure 7.3: Orbifold fixed points. The second **A**-torus can also be interpreted as a **B**-torus (cf. fig. 7.1 and sect. 2.2.2.2, p. 40- 42).

The remaining twelve 3-cycles arise in the $\mathbb{Z}_2$ twisted sector of the orbifold. Since Θ^2 acts non-trivially only onto the first two T^2, in the $\mathbb{Z}_2$ twisted sector the sixteen $\mathbb{Z}_2$ fixed points do appear as shown in figure 7.3.

The unfilled boxes in the figure indicate the $\mathbb{Z}_2$ fixed points which are also fixed under the $\mathbb{Z}_4$ symmetry. After blowing up the orbifold singularities, each of these fixed points gives rise an exceptional 2-cycle e_{ij} with the topology of S^2. These exceptional 2-cycles can be combined with the two fundamental 1-cycles on the third torus to form what might be called exceptional 3-cycles with the topology $S^2 \times S^1$. However, we have to take into account the $\mathbb{Z}_4$ action, which leaves four fixed points invariant and arranges the remaining twelve in six pairs. Since the $\mathbb{Z}_4$ acts by reflection on the third torus, its action on the exceptional cycles $e_{ij} \otimes \pi_{5,6}$ is

$$\Theta\left(e_{ij} \otimes \pi_{5,6}\right) = -e_{\theta(i)\theta(j)} \otimes \pi_{5,6} \tag{7.11}$$

with

$$\theta(1) = 1, \quad \theta(2) = 3, \quad \theta(3) = 2, \quad \theta(4) = 4 \tag{7.12}$$

Due to the minus sign in (7.11) the invariant $\mathbb{Z}_4$ fixed points drop out and what remains are precisely the twelve 3-cycles

$$\begin{aligned}
\varepsilon_1 &\equiv (e_{12} - e_{13}) \otimes \pi_5, & \bar{\varepsilon}_1 &\equiv (e_{12} - e_{13}) \otimes \pi_6 \\
\varepsilon_2 &\equiv (e_{42} - e_{43}) \otimes \pi_5, & \bar{\varepsilon}_2 &\equiv (e_{42} - e_{43}) \otimes \pi_6 \\
\varepsilon_3 &\equiv (e_{21} - e_{31}) \otimes \pi_5, & \bar{\varepsilon}_3 &\equiv (e_{21} - e_{31}) \otimes \pi_6 \\
\varepsilon_4 &\equiv (e_{24} - e_{34}) \otimes \pi_5, & \bar{\varepsilon}_4 &\equiv (e_{24} - e_{34}) \otimes \pi_6 \\
\varepsilon_5 &\equiv (e_{22} - e_{33}) \otimes \pi_5, & \bar{\varepsilon}_5 &\equiv (e_{22} - e_{33}) \otimes \pi_6 \\
\varepsilon_6 &\equiv (e_{23} - e_{32}) \otimes \pi_5, & \bar{\varepsilon}_6 &\equiv (e_{23} - e_{32}) \otimes \pi_6
\end{aligned} \tag{7.13}$$

Utilizing (7.9) the resulting intersection form is simply

$$I_\varepsilon = \bigoplus_{i=1}^{6} \begin{pmatrix} 0 & -2 \\ 2 & 0 \end{pmatrix} \tag{7.14}$$

These 3-cycles lie in $H_3(\mathcal{X}, \mathbb{Z})$ but do not form an integral basis of the free module since their intersection form is not uni-modular.

7.2.3 An integral basis of 3-cycles

The cycles which are missing so far are the ones corresponding to what is called fractional D-branes [29, 207]. In our context these are D-branes wrapping only one-half times around the toroidal cycles $\{\rho_1, \bar{\rho}_1, \rho_2, \bar{\rho}_2\}$ while wrapping simultaneously around some of the exceptional 3-cycles. Therefore in the orbifold limit such branes are stuck at the fixed points and one needs at least two such fractional D-branes in order to form a brane which can be moved into the bulk.

To proceed, we need a rule of what combinations of toroidal and exceptional cycles are allowed for a fractional D-brane. Such a rule can be easily gained from our physical intuition. A D-brane wrapping for instance the toroidal cycle $\frac{1}{2}\rho_1$ can only wrap around those exceptional 3-cycles that correspond to the $\mathbb{Z}_2$ fixed points the flat D-brane is passing through. In our case, when the brane is lying along the $X_{1,3,5}$-axis on the three T^2s, the allowed exceptional cycles are $\{\varepsilon_1, \varepsilon_3, \varepsilon_5\}$. Therefore, the total homological cycle the D-brane is wrapping on can be for instance

$$\pi_a = \frac{1}{2}\rho_1 + \frac{1}{2}(\varepsilon_1 + \varepsilon_3 + \varepsilon_5) \tag{7.15}$$

The relative signs for the four different terms in (7.15) are still free parameters and at the orbifold point do correspond to turning on a discrete Wilson line along a longitudinal internal direction of the D-brane. Note, that this construction is completely analogous to the construction of boundary states for fractional D-branes [208, 209, 210] carrying also a charge under some $\mathbb{Z}_2$ twisted sector states.

As an immediate consequences of this rule, only unbarred respectively barred cycles can be combined into fractional cycles, as they wrap the same fundamental 1-cycle on the third T^2. Apparently, the only non-vanishing intersection numbers are between barred and unbarred cycles. Any unbarred fractional D-brane can be expanded as

$$\pi_a = v_{a,1}\rho_1 + v_{a,2}\rho_2 + \sum_{i=1}^{6} v_{a,i+2}\, \varepsilon_i \tag{7.16}$$

with half-integer valued coefficients $v_{a,i}$. By exchanging the two fundamental cycles on the third T^2, we can associate to it a barred brane

$$\bar{\pi}_a = v_{a,1}\bar{\rho}_1 + v_{a,2}\bar{\rho}_2 + \sum_{i=1}^{6} v_{a,i+2}\, \bar{\varepsilon}_i \tag{7.17}$$

with the same coefficients $v_{a,i+8} = v_{a,i}$ for $i \in \{1, \dots, 8\}$. Using our rule we can construct all linear combinations with "self"-intersection number $\pi \circ \bar{\pi} = -2$, where we also have to keep in mind that the cycles form a lattice, i.e. integer linear combinations of cycles are again cycles.
In the following we list all the fractional 3-cycles with "self"-intersection number $\pi \circ \bar{\pi} = -2$. These cycles can be divided into 3 sets:

- **a)** $\{(v_1, v_2; v_3, v_4; v_5, v_6; v_7, v_8) \mid v_1 + v_2 = \pm 1/2,\ v_3 + v_4 = \pm 1/2,\ v_5 + v_6 = \pm 1/2,\ v_7 + v_8 = \pm 1/2;\ v_1 + v_3 + v_5 + v_7 = 0 \bmod 1\}$. These combinations are obtained by observing which fixed points the flat branes parallel to the fundamental cycles do intersect. These define $8 \cdot 16 = 128$ different fractional 3-cycles.

b) $\{(v_1, v_2; v_3, v_4; 0, 0; 0, 0),\ (v_1, v_2; 0, 0; v_5, v_6; 0, 0),\ (v_1, v_2; 0, 0; 0, 0; v_7, v_8),$ $(0, 0; v_3, v_4; v_5, v_6; 0, 0),\ (0, 0; v_3, v_4; 0, 0; v_7, v_8),\ (0, 0; 0, 0; v_5, v_6; v_7, v_8)$ $\mid v_i \in \pm 1/2\}\}$. The first three kinds of cycles are again constructed from branes lying parallel to the x,y-axis on one T^2 and stretching along the diagonal on the other T^2. The remaining three kinds of cycles arise from integer linear combinations of the cycles introduced so far. Thus, in total this yields $6 \cdot 16 = 96$ 3-cycles in the second set.

c) $\{(v_1, v_2; v_3, v_4; v_5, v_6; v_7, v_8) \mid$ exactly one $v_i = \pm 1$, rest zero$\}$. Only the vectors with $v_1 = \pm 1$ or $v_2 \pm 1$ can be derived from untwisted branes. They are purely untwisted. The purely twisted ones again arise from linear combinations. This third set contains $2 \cdot 8 = 16$ 3-cycles.

Altogether there are 240 of such 3-cycles with "self"-intersection number -2, which intriguingly just corresponds to the number of roots of the E_8 Lie algebra. Now, it is easy to write a computer program searching for a basis among these 240 cycles, so that the intersection form takes the following form

$$I = \begin{pmatrix} 0 & C_{E_8} \\ -C_{E_8} & 0 \end{pmatrix} \tag{7.18}$$

where C_{E_8} denotes the Cartan matrix of E_8

$$C_{E_8} = \begin{pmatrix} -2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & -2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -2 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & -2 \end{pmatrix} \tag{7.19}$$

One possible choice for the “simple roots” is

$$\begin{aligned}
\vec{w}_1 &= \frac{1}{2}(-1,\ \ 0,-1,\ \ 0,-1,\ \ 0,-1,\ \ 0) \\
\vec{w}_2 &= \frac{1}{2}(\ \ 1,\ \ 0,\ \ 1,\ \ 0,\ \ 1,\ \ 0,-1,\ \ 0) \\
\vec{w}_3 &= \frac{1}{2}(\ \ 1,\ \ 0,-1,\ \ 0,-1,\ \ 0,\ \ 1,\ \ 0) \\
\vec{w}_4 &= \frac{1}{2}(-1,\ \ 0,\ \ 1,\ \ 0,\ \ 0,\ \ 1,\ \ 0,\ \ 1) \\
\vec{w}_5 &= \frac{1}{2}(\ \ 0,\ \ 1,-1,\ \ 0,\ \ 1,\ \ 0,\ \ 0,-1) \\
\vec{w}_6 &= \frac{1}{2}(\ \ 0,-1,\ \ 1,\ \ 0,-1,\ \ 0,\ \ 0,-1) \\
\vec{w}_7 &= \frac{1}{2}(\ \ 0,\ \ 1,\ \ 0,\ \ 1,\ \ 0,-1,\ \ 0,\ \ 1) \\
\vec{w}_8 &= \frac{1}{2}(\ \ 0,-1,\ \ 0,-1,\ \ 0,-1,\ \ 0,\ \ 1)
\end{aligned} \tag{7.20}$$

Since the Cartan matrix is unimodular, we indeed have constructed an integral basis for the homology lattice $H_3(\mathcal{X},\mathbb{Z})$. In the following, it turns out to be more convenient to work with the non-integral orbifold basis allowing also half-integer coefficients. However, as we have explained not all such cycles are part of $H_3(\mathcal{X},\mathbb{Z})$, so we have to ensure each time we use such fractional 3-cycles that they are indeed contained in the unimodular lattice $H_3(\mathcal{X},\mathbb{Z})$, i.e. that they are integer linear combinations of the basis (7.20).

7.3 Orientifolds of the $\mathbb{Z}_4$ Type IIA orbifold

Equipped with the necessary information about the 3-cycles in the $\mathbb{Z}_4$ toroidal orbifold, we can move forward and consider the four inequivalent orientifold models in more detail.

7.3.1 O6-planes in the $\mathbb{Z}_4$ orientifold

First, we have to determine the 3-cycle of the O6-planes. Let us discuss this computation for the **ABB** model in some more detail, as this orientifold will be of main interest for its potential to provide semi-realistic standard-like models.

We have to determine the fixed point sets of the four relevant orientifold projections $\{\Omega\bar{\sigma}, \Omega\bar{\sigma}\Theta, \Omega\bar{\sigma}\Theta^2, \Omega\bar{\sigma}\Theta^3\}$. The results are listed in table 7.2. Adding up all contributions we get

$$\begin{aligned}
\pi_{\mathrm{O6}} &= 4\,\pi_{145} + 4\,\pi_{235} + 4\,\pi_{136} - 4\,\pi_{246} - 4\,\pi_{146} - 4\,\pi_{236} \\
&= 2\,\rho_2 + 2\,\bar{\rho}_1 - 2\,\bar{\rho}_2.
\end{aligned} \tag{7.21}$$

Thus, only bulk cycles appear in π_{O6} reflecting the fact that in the conformal field theory the orientifold planes carry only charge under untwisted R-R fields [2,57]. The next step is to determine the action of $\Omega\bar{\sigma}$ on the homological cycles.

Projection	fixed point set
$\Omega\,\bar\sigma$	$2\,\pi_{135} + 2\,\pi_{145}$
$\Omega\,\bar\sigma\,\Theta$	$2\,\pi_{145} + 2\,\pi_{245} - 4\,\pi_{146} - 4\,\pi_{246}$
$\Omega\,\bar\sigma\,\Theta^2$	$2\,\pi_{235} - 2\,\pi_{245}$
$\Omega\,\bar\sigma\,\Theta^3$	$-2\,\pi_{135} + 2\,\pi_{235} + 4\,\pi_{136} - 4\,\pi_{236}$

Table 7.2: O6-planes for **ABB** model

This can easily be done for the orbifold basis. We find for the toroidal 3-cycles

$$\begin{aligned} &\rho_1 \to \rho_2, \qquad &&\bar\rho_1 \to \rho_2 - \bar\rho_2 \\ &\rho_2 \to \rho_1, &&\bar\rho_2 \to \rho_1 - \bar\rho_1 \end{aligned} \tag{7.22}$$

and for the exceptional cycles

$$\begin{aligned} &\varepsilon_1 \to -\varepsilon_1 \qquad &&\bar\varepsilon_1 \to -\varepsilon_1 + \bar\varepsilon_1 \\ &\varepsilon_2 \to -\varepsilon_2 &&\bar\varepsilon_2 \to -\varepsilon_2 + \bar\varepsilon_2 \\ &\varepsilon_3 \to \varepsilon_3 &&\bar\varepsilon_3 \to \varepsilon_3 - \bar\varepsilon_3 \\ &\varepsilon_4 \to \varepsilon_4 &&\bar\varepsilon_4 \to \varepsilon_4 - \bar\varepsilon_4 \\ &\varepsilon_5 \to \varepsilon_6 &&\bar\varepsilon_5 \to \varepsilon_6 - \bar\varepsilon_6 \\ &\varepsilon_6 \to \varepsilon_5 &&\bar\varepsilon_6 \to \varepsilon_5 - \bar\varepsilon_5 \end{aligned} \tag{7.23}$$

Consistently, the orientifold plane (7.21) is invariant under the $\Omega\bar\sigma$ action. For the other three orientifold models, the results for the O6 planes and the action of $\Omega\bar\sigma$ on the homology lattice can be found in appendix D.1. In principle, we have now provided all the information that is necessary to build intersecting brane world models on the $\mathbb{Z}_4$ orientifold. However, since we are particularly interested in supersymmetric models we need to have control not only over topological data of the D6-branes but over the nature of the sLag cycles as well.

7.3.2 Supersymmetric cycles

The metric at the orbifold point is flat up to some isolated orbifold singularities. Therefore, flat D6-branes in a given homology class are definitely special Lagrangian. We restrict our D6-branes to be flat and factorizable in the sense that they can be described by six wrapping numbers, (n_I, m_I) with $I = 1, 2, 3$, along the fundamental toroidal cycles, where for each I the integers (n_I, m_I) are relatively co-prime. Given such a bulk brane, one can compute the homology class that it wraps expressed in the $\mathbb{Z}_4$ basis

$$\begin{aligned} \pi_a^{\text{bulk}} = &[(n_{a,1}\,n_{a,2} - m_{a,1}\,m_{a,2})n_{a,3}]\ \rho_1 + [(n_{a,1}\,m_{a,2} + m_{a,1}\,n_{a,2})n_{a,3}]\ \rho_2 \\ &+ [(n_{a,1}\,n_{a,2} - m_{a,1}\,m_{a,2})m_{a,3}]\ \bar\rho_1 + [(n_{a,1}\,m_{a,2} + m_{a,1}\,n_{a,2})m_{a,3}]\ \bar\rho_2 \end{aligned} \tag{7.24}$$

For the **ABB** orientifold, the condition that such a D6-brane preserves the same supersymmetry as the orientifold plane is simply

$$\varphi_{a,1} + \varphi_{a,2} + \varphi_{a,3} = \frac{\pi}{4} \mod 2\pi \tag{7.25}$$

with

$$\tan\varphi_{a,1} = \frac{m_{a,1}}{n_{a,1}}, \quad \tan\varphi_{a,2} = \frac{m_{a,2}}{n_{a,2}}, \quad \tan\varphi_{a,3} = \frac{U_2\, m_{a,3}}{n_{a,3} + \frac{1}{2} m_{a,3}} \tag{7.26}$$

Taking the tan(...) on both sides of equation (7.25) we can reformulate the supersymmetry condition in terms of wrapping numbers (Note, that this only yields a necessary condition as tan(...) is just periodic mod π.)

$$U_2 = \frac{\left(n_{a,3} + \frac{1}{2} m_{a,3}\right)}{m_{a,3}} \frac{\left(n_{a,1}\, n_{a,2} - m_{a,1}\, m_{a,2} - n_{a,1}\, m_{a,2} - m_{a,1}\, n_{a,2}\right)}{\left(n_{a,1}\, n_{a,2} - m_{a,1}\, m_{a,2} + n_{a,1}\, m_{a,2} + m_{a,1}\, n_{a,2}\right)} \tag{7.27}$$

Therefore, the complex structure of the third torus in general is already fixed by one supersymmetric D-brane. In case one introduces more D6-branes, one gets non-trivial conditions on the wrapping numbers of these D-branes. The supersymmetry conditions for the other three orientifold models are summarized in appendix D.2 (p. 218).

Working only with the bulk branes (7.24), the model building possibilities are very restricted. In particular, it seems to be impossible to get large enough gauge groups to accommodate the Standard Model gauge symmetry, $U(3) \times U(2) \times U(1)$, of at least rank six. One such supersymmetric model with only bulk branes and rank four has been constructed in [43]. Now, to enlarge the number of possibilities, we also allow such flat, factorizable branes to pass through $\mathbb{Z}_2$ fixed points and split into fractional D-branes. Thus, according to our rule we allow fractional D-branes wrapping the cycle

$$\pi_a^{\text{frac}} = \frac{1}{2}\pi_a^{\text{bulk}} + \frac{n_{a,3}}{2}\left[\sum_{j=1}^{6} w_{a,j}\varepsilon_j\right] + \frac{m_{a,3}}{2}\left[\sum_{j=1}^{6} w_{a,j}\bar{\varepsilon}_j\right] \tag{7.28}$$

with $w_{a,j} \in \{0, \pm 1\}$. To make contact with the formerly introduced coefficients $v_{a,j}$, we define

$$v_{a,j} = \frac{n_{a,3}}{2}\, w_{a,j}, \qquad v_{a,j+8} = \frac{m_{a,3}}{2}\, w_{a,j} \tag{7.29}$$

for $j \in \{1, \ldots, 8\}$. In (7.28) we have taken into account that the $\mathbb{Z}_2$ fixed points all lie on the first two two-dimensional tori and that on the third torus fractional D-branes do have winding numbers along the two fundamental 1-cycles. Moreover, since ε_j and $\bar{\varepsilon}_j$ only differ by the cycle on the third torus, their coefficients in (7.28) must indeed be equal.

These fractional D6-branes do correspond to the following boundary states in the conformal field theory of the $T^6/\mathbb{Z}_4$ orbifold model

$$
\begin{aligned}
&\Big| D^{\rm frac}; (n_I, m_I), \alpha_{ij} \Big\rangle = \\
&\frac{1}{4\sqrt{2}} \left(\prod_{j=1}^{2} \sqrt{n_j^2 + m_j^2} \right) \sqrt{n_3^2 + n_3 m_3 + \tfrac{m_3^2}{2}} \Big(\big| D; (n_I, m_I) \big\rangle_U + \big| D; \Theta(n_I, m_I) \big\rangle_U \Big) \\
&\quad + \frac{1}{2\sqrt{2}} \sqrt{n_3^2 + n_3 m_3 + \tfrac{m_3^2}{2}} \\
&\qquad \times \left(\sum_{i,j=1}^{4} \alpha_{ij} \big| D; (n_I, m_I), e_{ij} \big\rangle_T + \sum_{i,j=1}^{4} \alpha_{ij} \big| D; \Theta(n_I, m_I), \Theta(e_{ij}) \big\rangle_T \right)
\end{aligned}
\tag{7.30}
$$

In the schematic form of the boundary state (7.30) there are contributions from both the untwisted and the $\mathbb{Z}_2$ twisted sector and we have taken the orbit under the $\mathbb{Z}_4$ symmetry Θ with the following action on the winding numbers

$$
\Theta(n_{1,2}, m_{1,2}) = (-m_{1,2}, n_{1,2}), \qquad \Theta(n_3, m_3) = -(n_3, m_3) \tag{7.31}
$$

implying that Θ^2 acts like the identity on the boundary states. This explains why only two and not four untwisted boundary states do appear in (7.30). Note, that in the sum over the $\mathbb{Z}_2$ fixed points, for each D6-brane precisely four coefficients take values $\alpha_{ij} \in \{-1, +1\}$ and the remaining ones are vanishing. The α_{ij} are of course directly related to the coefficients w_i appearing in the description of the corresponding fractional 3-cycles. For the interpretation of these coefficients α_{ij}, one has to remember that changing the sign of α_{ij} corresponds to turning on a discrete $\mathbb{Z}_2$ Wilson line along one internal direction of the brane [208, 210]. The action of Θ on the twisted sector ground states e_{ij} is the same as in (7.11). The elementary boundary states like $|D; (n_I, m_I)\rangle_U$ are the usual ones for a flat D6 brane with wrapping numbers (n_I, m_I) on $T^6 = T^2 \times T^2 \times T^2$ and can be found in Appendix C. The important normalization factors in (7.30) are fixed by the Cardy condition (cf. [211]), stating that the result for the annulus partition function must coincide for the loop- and the tree-channel computation.

Since the brane and its $\mathbb{Z}_4$ image only break the supersymmetry down to $\mathcal{N} = 2$, one gets a $\mathcal{N} = 2$ $U(N)$ vector multiplet on each stack of fractional D-branes. The scalars in these vector multiplets correspond to the position of the D6-brane on the third T^2 torus, which is still an open string modulus.

Coming back to the homology cycles, following our general rule for fractional branes imposes further constraints on the coefficients because only those exceptional cycles are allowed to contribute which are intersected by the flat D-brane. The only allowed exceptional 3-cycles are summarized in table 7.3, depending on the wrapping numbers of the first two tori T^2. At first glance, there is a mismatch between the number of parameters describing a 3-cycle and the corresponding boundary state. For each D6-brane there are three non-vanishing parameters w_i but four α_{ij}. However, a flat fractional brane and its $\mathbb{Z}_4$ image always intersect in precisely one $\mathbb{Z}_4$ fixed point times a circle on the third T^2.

	n_1 odd, m_1 odd	n_1 odd, m_1 even	n_1 even, m_1 odd
n_2 odd m_2 odd		ε_3, ε_4 ε_5, ε_6	ε_3, ε_4 ε_5, ε_6
n_2 odd m_2 even	ε_1, ε_2 ε_5, ε_6	ε_1, ε_3, ε_5 ε_1,ε_4, ε_6 ε_2,ε_3, ε_6 ε_2,ε_4, ε_5	ε_1,ε_3, ε_6 ε_1,ε_4, ε_5 ε_2,ε_3, ε_5 ε_2,ε_4, ε_6
n_2 even m_2 odd	ε_1, ε_2 ε_5, ε_6	ε_1, ε_3, ε_6 ε_1, ε_4, ε_5 ε_2, ε_3, ε_5 ε_2, ε_4, ε_6	ε_1, ε_3, ε_5 ε_1, ε_4, ε_6 ε_2, ε_3, ε_6 ε_2, ε_4, ε_5

Table 7.3: Allowed exceptional cycles

Since Θ acts on this fixed locus with a minus sign, this twisted sector effectively drops out of the boundary state (7.30). A different way of saying this is that at the intersection between the brane and its $\mathbb{Z}_4$ image, there lives a hyper-multiplet, Φ_{adj}, in the adjoint representation. Since it is an $\mathcal{N}=2$ supermultiplet, there exists a flat direction in the D-term potential corresponding to the recombination of the two branes into a single brane. This single brane of course no longer runs to the $\mathbb{Z}_4$ invariant fixed point. This brane recombination process is depicted in figure 7.4.

A non-trivial test for our considerations is the condition that a fractional brane (7.28) transformed to the E_8-basis must have integer coefficients. To see this, we write the 8×8 matrix (7.19) and a second identical copy as the two diagonal blocks of a 16×16 matrix, and then act with the inverse of the transposed matrix onto a general vector (7.28). Then we have to investigate the different cases according to table 7.3 separately. For instance for the case n_1 odd, n_2 odd, m_1 even, m_2 odd and fractional cycles ε_3, ε_4 with signs w_3, w_4 respectively, we substitute $m_1 = 2k_1$ and obtain the following vector in the E_8-basis:

$$\left[\Big(\frac{1}{2}(n_1m_2 - w_3) + k_1n_2\Big)n_3,\ \Big(\frac{1}{2}(n_1n_2 - w_3) - k_1m_2 + n_1m_2 + 2k_1n_2\Big)n_3, \ldots\right] \tag{7.32}$$

Already for the first two components we can see what generally happens for all cases and components: since n_1, n_2, m_2 and w_3 are non-vanishing and because products of odd numbers are also odd, just sums and differences of two odd numbers occur and these are always even or zero and therefore can be divided by 2 and still lead to integer coefficients. Having defined a well understood set of supersymmetric fractional D6-branes, we are now in the position to search for phenomenologically interesting supersymmetric intersecting brane worlds.

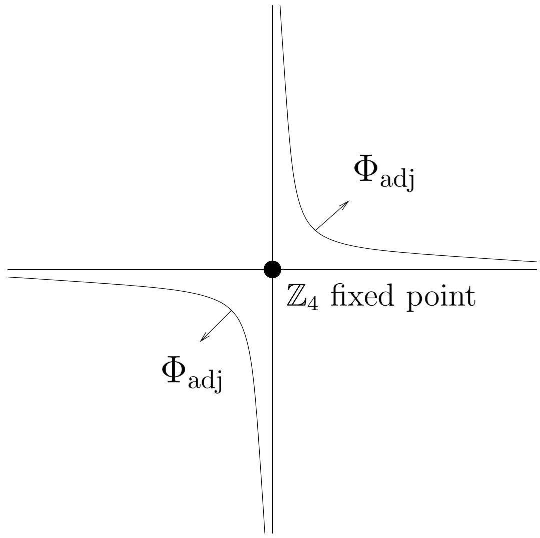


Figure 7.4: Recombined branes

7.4 A four generation supersymmetric Pati-Salam model

In this section we present the construction of a semi-realistic supersymmetric intersecting brane world model. This provides an application of the formalism developed in the previous sections. It turns out that the **ABB** orientifold model is the most appropriate one for doing this. Using the fractional D6-branes introduced in the last section, one finds that by requiring that no (anti-) symmetric representations of the $U(N_a)$ gauge groups do appear, only very few sufficiently small mutual intersection numbers arise. For the **ABB** model with the complex structure of the last torus being $U_2 = 1$, an extensive computer search reveals that essentially only mutual intersection numbers $(\pi_a \circ \pi_b, \pi'_a \circ \pi_b) = (0,0), (\pm 2, \mp 2)$ are possible. Even with these intersection numbers it is possible to construct a four generation supersymmetric Pati-Salam model with initial gauge group $U(4) \times U(2) \times U(2)$. A typical model of this sort can be realized by the three stacks of D6-branes presented in table 7.4.

Computing the intersection numbers for these D6-branes and using the general formula for the chiral massless spectrum, one gets the massless modes shown in table 7.5. Here we have normalized as usual the gauge fields in the diagonal $U(1)_a \subset U(N_a)$ sub-algebras as

$$A^\mu_{U(1)_a} = \frac{1}{N_a} \operatorname{Tr}\left(A^\mu_{U(N_a)}\right) \tag{7.33}$$

Note, that all non-abelian gauge anomalies are canceled. Adding up all homological cycles, one finds that the RR-tadpole cancellation condition (7.1) is indeed satisfied. A nice check is whether the NS-NS tadpole cancellation

Stack	(n_I, m_I)	Homology cycle
U(4)	$(-1,0;1,1;-1,0)$	$\pi_1 = \frac{1}{2}(\rho_1 + \rho_2 - \varepsilon_5 + \varepsilon_6)$ $\pi_1' = \frac{1}{2}(\rho_1 + \rho_2 + \varepsilon_5 - \varepsilon_6)$
U(2)	$(0,1;-1,-1;-1,2)$	$\pi_2 = \frac{1}{2}(-\rho_1 + \rho_2 + 2\bar\rho_1 - 2\bar\rho_2 + \varepsilon_5 - \varepsilon_6 - 2\bar\varepsilon_5 + 2\bar\varepsilon_6)$ $\pi_2' = \frac{1}{2}(-\rho_1 + \rho_2 + 2\bar\rho_1 - 2\bar\rho_2 + \varepsilon_5 - \varepsilon_6 - 2\bar\varepsilon_5 + 2\bar\varepsilon_6)$
U(2)	$(-1,0;1,-1;1,-2)$	$\pi_3 = \frac{1}{2}(-\rho_1 + \rho_2 + 2\bar\rho_1 - 2\bar\rho_2 - \varepsilon_5 + \varepsilon_6 + 2\bar\varepsilon_5 - 2\bar\varepsilon_6)$ $\pi_3' = \frac{1}{2}(-\rho_1 + \rho_2 + 2\bar\rho_1 - 2\bar\rho_2 - \varepsilon_5 + \varepsilon_6 + 2\bar\varepsilon_5 - 2\bar\varepsilon_6)$

Table 7.4: D6-branes for a 4 generation PS-model

n	$SU(4) \times SU(2) \times SU(2) \times U(1)^3$
2	$(4,2,1)_{(1,-1,0)}$
2	$(4,2,1)_{(1,1,0)}$
2	$(\bar{4},1,2)_{(-1,0,1)}$
2	$(\bar{4},1,2)_{(-1,0,-1)}$

Table 7.5: Chiral spectrum for 4 generation PS-model

condition (7.2) is also satisfied, as it should be for a globally supersymmetric configuration. For the contribution of the O6-plane to the scalar potential, one finds

$$V_{\rm O6} = -T_6\, e^{-\phi_4} 16\sqrt{2} \left(\frac{1}{\sqrt{U_2}} + 2\sqrt{U_2} \right) \tag{7.34}$$

whereas the three stacks of D6-branes give

$$\begin{aligned} V_1 &= T_6\, e^{-\phi_4} 16\sqrt{2} \frac{1}{\sqrt{U_2}} \\ V_{2,3} &= T_6\, e^{-\phi_4} 16\sqrt{2} \sqrt{U_2} \end{aligned} \tag{7.35}$$

We see that the scalar potential vanishes for all values of the complex structure U_2 of the third torus. Thus, the disc level scalar potential indeed vanishes and we have constructed a globally supersymmetric intersecting brane world model with gauge group $U(4) \times U(2) \times U(2)$.

7.4.1 Green-Schwarz mechanism

Computing in the usual way the mixed $U(1)_a - SU(N_b)^2$ anomalies, one confirms the general result derived in [43]

$$A_{ab} = \frac{N_a}{4}(-\pi_a + \pi'_a) \circ (\pi_b + \pi'_b) \tag{7.36}$$

In our example there is only one anomalous $U(1)$ while $U(1)_2$ and $U(1)_3$ are anomaly-free. This anomaly is canceled by some generalized Green-Schwarz mechanism involving the axionic couplings from the Chern-Simons terms in the effective action on the D6-branes [5]

$$S^F_{CS} = \sum_{i=1}^{b_3} \int d^4x\, N_a\, (v_{a,i} - v'_{a,i})\, B_i \wedge F_a \tag{7.37}$$

and

$$S^{F\wedge F}_{CS} = \sum_{i=1}^{b_3} \int d^4x\, (v_{b,i} + v'_{b,i})\, \Phi_i\, \mathrm{Tr}(F_b \wedge F_b) \tag{7.38}$$

where B_i is defined as the integral of the RR 5-form over the corresponding 3-cycle and similarly Φ_i is defined as the integral of the RR 3-form over the corresponding 3-cycle. Taking into account the Hodge duality between the fields B_i and Φ_{i+8} these axionic couplings indeed cancel the mixed anomalies. For more details we refer the reader to the general discussion in [43].

As was pointed out in [163] the couplings (7.37) can generate a mass term for $U(1)$ gauge fields even if they are not anomalous. The massless $U(1)$s are given by the kernel of the matrix

$$M_{ai} = N_a(v_{a,i} - v'_{a,i}) \tag{7.39}$$

In our model it can be easily seen that $U(1)_2$ and $U(1)_3$ remain massless, so that the final gauge symmetry is $SU(4) \times SU(2) \times SU(2) \times U(1)^2$. We will not discuss this model any further but move forward to the construction of a more realistic model with three generations.

7.5 Three generation supersymmetric Pati-Salam model

For the **ABB** model with the complex structure of the last torus fixed at $U_2 = 1/2$, a computer search shows that only sufficiently small mutual intersection numbers $(\pi_a \circ \pi_b, \pi'_a \circ \pi_b) = (0,0), (\pm 1, 0), (0, \pm 1)$ are possible. These numbers allow the construction of a three generation model in the following way. First, we start with seven stacks of D6-branes with an initial gauge symmetry $U(4) \times U(2)^6$ and choose the wrapping numbers as shown in table 7.6.

[5] The sign in front of $v'_{a,i}$ is due to the fact that $F_{a'} = -F_a$. Similarly this sign cancels in (7.38).

Stack	Homology cycle (ρ, ϵ-basis)
$U(4)_1$	$\pi_1 = \frac{1}{2}\left(\bar\rho_1 - \bar\varepsilon_1 - \bar\varepsilon_3 - \bar\varepsilon_5\right)$ $\pi_1' = \frac{1}{2}\left(\rho_2 - \bar\rho_2 + \varepsilon_1 - \varepsilon_3 - \varepsilon_6 - \bar\varepsilon_1 + \bar\varepsilon_3 + \bar\varepsilon_6\right)$
$U(2)_2$	$\pi_2 = \frac{1}{2}\left(\bar\rho_1 - \bar\varepsilon_1 + \bar\varepsilon_3 + \bar\varepsilon_5\right)$ $\pi_2' = \frac{1}{2}\left(\rho_2 - \bar\rho_2 + \varepsilon_1 + \varepsilon_3 + \varepsilon_6 - \bar\varepsilon_1 - \bar\varepsilon_3 - \bar\varepsilon_6\right)$
$U(2)_3$	$\pi_3 = \frac{1}{2}\left(\bar\rho_1 - \bar\varepsilon_2 + \bar\varepsilon_3 + \bar\varepsilon_6\right)$ $\pi_3' = \frac{1}{2}\left(\rho_2 - \bar\rho_2 + \varepsilon_2 + \varepsilon_3 + \varepsilon_5 - \bar\varepsilon_2 - \bar\varepsilon_3 - \bar\varepsilon_5\right)$
$U(2)_4$	$\pi_4 = \frac{1}{2}\left(\bar\rho_1 + \bar\varepsilon_2 + \bar\varepsilon_3 + \bar\varepsilon_6\right)$ $\pi_4' = \frac{1}{2}\left(\rho_2 - \bar\rho_2 - \varepsilon_2 + \varepsilon_3 + \varepsilon_5 + \bar\varepsilon_2 - \bar\varepsilon_3 - \bar\varepsilon_5\right)$
$U(2)_5$	$\pi_5 = \frac{1}{2}\left(\bar\rho_1 + \bar\varepsilon_1 - \bar\varepsilon_3 + \bar\varepsilon_5\right)$ $\pi_5' = \frac{1}{2}\left(\rho_2 - \bar\rho_2 - \varepsilon_1 - \varepsilon_3 + \varepsilon_6 + \bar\varepsilon_1 + \bar\varepsilon_3 - \bar\varepsilon_6\right)$
$U(2)_6$	$\pi_6 = \frac{1}{2}\left(\bar\rho_1 + \bar\varepsilon_1 + \bar\varepsilon_4 - \bar\varepsilon_6\right)$ $\pi_6' = \frac{1}{2}\left(\rho_2 - \bar\rho_2 - \varepsilon_1 + \varepsilon_4 - \varepsilon_5 + \bar\varepsilon_1 - \bar\varepsilon_3 + \bar\varepsilon_5\right)$
$U(2)_7$	$\pi_7 = \frac{1}{2}\left(\bar\rho_1 + \bar\varepsilon_1 - \bar\varepsilon_4 - \bar\varepsilon_6\right)$ $\pi_7' = \frac{1}{2}\left(\rho_2 - \bar\rho_2 - \varepsilon_1 - \varepsilon_4 - \varepsilon_5 + \bar\varepsilon_1 + \bar\varepsilon_3 + \bar\varepsilon_5\right)$

Table 7.6: D6-branes for 3 generation PS-model. All the branes have wrapping numbers $(n_I, m_I) = (1,0;1,0;0,1)$ for the untwisted part.

Adding up all homological 3-cycles, one realizes that the RR-tadpole cancellation condition is satisfied. The contribution of the O6-plane tension to the scalar potential is

$$V_{\rm O6} = -T_6\, e^{-\phi_4} 16\sqrt{2}\left(\frac{1}{\sqrt{U_2}} + 2\sqrt{U_2}\right) \tag{7.40}$$

whereas the seven stacks of D6-branes give

$$\begin{aligned} V_1 &= T_6\, e^{-\phi_4} 16\sqrt{\frac{1}{4\,U_2} + U_2} \\ V_{2,\ldots,7} &= T_6\, e^{-\phi_4} 8\sqrt{\frac{1}{4\,U_2} + U_2} \end{aligned} \tag{7.41}$$

Adding up all terms, one finds that indeed the NS-NS tadpole vanishes just for $U_2 = \frac{1}{2}$. This means that in contrast to the four generation model, here supersymmetry really fixes the complex structure of the third torus (if we assume that no curved sLags exist in a neighborhood of $U_2 = \frac{1}{2}$). This freezing of moduli for supersymmetric backgrounds is very similar to what happens for instance in recently discussed compactifications with non-vanishing R-R fluxes [212, 212].

In terms of $\mathcal{N} = 2$ super-multiplets, the model contains vector multiplets in the gauge group $U(4) \times U(2)^3 \times U(2)^3$ and in addition two hyper-multiplets in

Field	n	$U(4) \times U(2)^3 \times U(2)^3$
$\Phi_{1'2}$	1	$(4; 2, 1, 1; 1, 1, 1)$
$\Phi_{1'3}$	1	$(4; 1, 2, 1; 1, 1, 1)$
$\Phi_{1'4}$	1	$(4; 1, 1, 2; 1, 1, 1)$
$\Phi_{1'5}$	1	$(\bar{4}; 1, 1, 1; \bar{2}, 1, 1)$
$\Phi_{1'6}$	1	$(\bar{4}; 1, 1, 1; 1, \bar{2}, 1)$
$\Phi_{1'7}$	1	$(\bar{4}; 1, 1, 1; 1, 1, \bar{2})$
$\Phi_{2'3}$	1	$(1; \bar{2}, \bar{2}, 1; 1, 1, 1)$
$\Phi_{2'4}$	1	$(1; \bar{2}, 1, \bar{2}; 1, 1, 1)$
$\Phi_{3'4}$	1	$(1; 1, \bar{2}, \bar{2}; 1, 1, 1)$
$\Phi_{5'6}$	1	$(1; 1, 1, 1; 2, 2, 1)$
$\Phi_{5'7}$	1	$(1; 1, 1, 1; 2, 1, 2)$
$\Phi_{6'7}$	1	$(1; 1, 1, 1; 1, 2, 2)$

Table 7.7: Chiral spectrum for a 7-stack model

the adjoint representation of each unitary gauge factor. The complex scalar in the vector multiplet corresponds to the unconstrained position of each stack of D6-branes on the third T^2. As described at the end of section 7.3.2, the hyper-multiplet appears on the intersection between a stack of branes and its $\mathbb{Z}_4$ image. By computing the intersection numbers, we derive the chiral spectrum as shown in table 7.7, where n denotes the number of chiral multiplets in the respective representation as given by the intersection number.

First, we notice that all non-abelian anomalies cancel including formally also the $U(2)$ anomalies.

In order to proceed and really get a three generation model, it is necessary to break the two triplets $U(2)^3$ down to their diagonal subgroups. Potential gauge symmetry breaking candidates in this way are the chiral fields $\{\Phi_{2'3}, \Phi_{2'4}, \Phi_{3'4}\}$ and $\{\Phi_{5'6}, \Phi_{5'7}, \Phi_{6'7}\}$ from table 7.7. However, one has to remember that these are chiral $\mathcal{N} = 1$ super-multiplets living on the intersection of two D-branes in every case. Let us review what massless bosons localized on intersecting D-branes indicate.

7.5.1 Brane recombination

If two stacks of D-branes preserve a common $\mathcal{N} = 2$ supersymmetry, then a massless hyper-multiplet, H, localized on the intersection, signals a possible deformation of the two stacks of D-branes into recombined D-branes which wrap a complex cycle. Note, that two factorizable branes can only preserve $\mathcal{N} = 2$ supersymmetry if they are parallel on one of the three T_I^2 tori. The complex cycle has the same volume as the sum of volumes of the two D-branes before the recombination process occurs. In the effective low energy theory, this recombination can be understood as a Higgs effect where a flat direction

$\langle h_1 \rangle = \langle h_2 \rangle$ in the D-term potential

$$V_D = \frac{1}{2g^2} \left(h_1 \bar{h}_1 - h_2 \bar{h}_2 \right)^2 \tag{7.42}$$

exists, along which the $U(N) \times U(N)$ gauge symmetry is broken to the diagonal subgroup.[6] Here h_1 and h_2 denote the two complex bosons inside the hypermultiplet. Thus, in this case without changing the closed string background, there exists an open string modulus, which has the interpretation of a Higgs field in the low energy effective theory. Note, that in the T-dual picture, this is just the deformation of a small instanton into an instanton of finite size. In our concrete models such $\mathcal{N} = 2$ Higgs sectors are coupled at brane intersections to chiral $\mathcal{N} = 1$ sectors. Note, that the brane recombination in the effective gauge theory cannot simply be described by the renormalizable couplings. In order to get the correct light spectrum, one also has to take into account stringy higher dimensional couplings.

When the two D-branes only preserve $\mathcal{N} = 1$ supersymmetry and support a massless chiral super-multiplet Φ on the intersection [213, 101], the situation gets a little bit more involved. In this case, the analogous D-term potential is of the form

$$V_D = \frac{1}{2g^2} \left(\phi \bar{\phi} \right)^2 \tag{7.43}$$

which tells us that, unless there are more chiral fields involved, simply by giving a VEV to the massless boson ϕ, we do not obtain a flat direction of the D-term potential and therefore break supersymmetry. Nevertheless, the massless modes indicate that the intersecting brane configuration lies on a line of marginal stability in the complex structure moduli space. By a small variation of the complex structure, a Fayet-Iliopoulos (FI) term, r, is introduced that changes the D-term potential to

$$V_D = \frac{1}{2g^2} \left(\phi \bar{\phi} + r \right)^2 \tag{7.44}$$

Therefore, for $r < 0$ the field ϕ becomes tachyonic and there exists a new stable supersymmetric minimum of the D-term potential. The intersecting branes then have combined into one D-brane wrapping a special Lagrangian 3-cycle in the underlying Calabi-Yau. For a finite FI-term r, this 3-cycle has smaller volume than the two intersecting branes. However, the two volumes are precisely equal on the line of marginal stability. This means that on the line of marginal stability, there exists a different configuration with only a single brane which also preserves the same $\mathcal{N} = 1$ supersymmetry and has the same volume as the former pair of intersecting D-branes. Again the gauge symmetry is broken to the diagonal subgroup. It has to be emphasized that in this case the two configurations are not simply linked by a Higgs mechanism in the effective low energy gauge theory. As mentioned before, in order to deform the intersecting brane configuration into the non-flat D-brane wrapping a special Lagrangian 3-cycle,

[6] If on one of the two stacks there sits only a single D6-brane, the F-term potential $\phi h_1 h_2$ forbids the existence of a flat direction with $\langle h_1 \rangle = \langle h_2 \rangle$. This is the field theoretic correspondence of the fact that there do not exist large instantons in the $U(1)$ gauge group. We thank A. Uranga for pointing this out to us.

one first has to deform the closed string background and then let the tachyonic mode condense. Therefore, the description of this process is intrinsically stringy and should be better described by string field theory rather than the effective low energy gauge theory[7]. For $r > 0$, the non-supersymmetric intersecting branes are stable and have a smaller volume than the recombined brane. The lift of these brane recombination processes to M-theory was discussed in [214].

After this little excursion, we come back to our model. We have seen that the condensation of hyper-multiplets is under much better control than the condensation of chiral multiplets. Therefore, we have to determine the Higgs fields in our model as well, meaning to compute the non-chiral spectrum. This cannot be done by a simple homology computation, but fortunately we do know the exact conformal field theory at the orbifold point. Using the boundary states (7.30), we can determine the non-chiral matter living on intersections of the various stacks of D-branes. One first computes the overlap between two such boundary states and then transforms the result to the open string channel to get the annulus partition function, from which one can read off the massless states. This is a straightforward but tedious computation, which also confirms the chiral spectrum in table 7.7. Thus the conformal field theory result agrees completely with the purely topological computation of the intersection numbers.

Computing the non-chiral spectrum just for one stack of $U(2)$ branes and their $\mathbb{Z}_4$- and $\Omega\bar{\sigma}$-images, one first finds the well known hyper-multiplet, $\Phi_{\rm adj} = (\phi_{\rm adj}, \tilde{\phi}_{\rm adj})$, in the adjoint representation of $U(2)$ localized on the intersection of a brane and its $\mathbb{Z}_4$ image. Moreover, there are two chiral multiplets, Ψ_A and $\Psi_{\bar{A}}$, in the $\mathbf{A}$ respectively $\bar{\mathbf{A}}$ representation arising from the (π_i, π'_i) sector. Since the two chiral fields carry conjugate representations of the gauge group, they cannot be seen by the topological intersection number which in fact vanishes, $\pi_i \circ \pi'_i = 0$. We have depicted the resulting quiver diagram for these three fields in figure 7.5. For each closed polygon in the quiver diagram (respecting the directions of the arrows), the associated product of fields can occur in the holomorphic super-potential. In our case, the following two terms can appear

$$W = \phi_{\rm adj}\Psi_A\Psi_{\bar{A}} + \tilde{\phi}_{\rm adj}\Psi_A\Psi_{\bar{A}} \tag{7.45}$$

which generate a mass for the anti-symmetric fields when the adjoint multiplet gets a VEV. As we have mentioned already in the last section, giving a VEV to this adjoint field localized on the intersection between a brane and its $\mathbb{Z}_4$ image, leads to the recombination of these two branes. The recombined brane no longer passes through the $\mathbb{Z}_4$ invariant intersection points. After computing

[7] In the context of so-called quasi-supersymmetric intersecting brane world models [186] , it has been observed that indeed the brane recombination of $\mathcal{N} = 1$ supersymmetric intersections cannot simply be described by a Higgs mechanism of massless modes. It was suggested there that the stringy nature of this transition has the meaning that also some massive, necessarily non-chiral, fields are condensing during the brane recombination. At least from the effective gauge theory point of view, this could induce the right mass terms which are necessary for an understanding of the new massless modes after the recombination. We leave it for future work to find the right effective description of this transition, but we can definitely state that it must involve some stringy aspects as the complex structure changes, i.e. the closed string background.

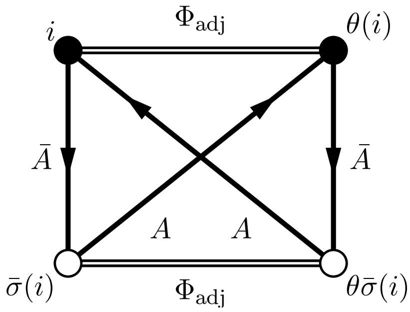


Figure 7.5: Adjoint higgsing

all annulus partition functions for pairs of D-branes from table 7.6, we find the total non-chiral spectrum listed in table 7.8.

It is interesting that we find Higgs fields which might break a $SU(4) \times SU(2) \times SU(2)$ gauge symmetry in a first step down to the Standard Model and in a second step down to $SU(3) \times U(1)_{em}$. However, the Higgs fields which would allow us to break the product groups $U(2)^3$ down to their diagonal subgroup are not present in the non-chiral spectrum.

7.5.2 D-flatness

However, we do have the massless chiral bifundamental fields $\{\Phi_{2'3}, \ldots, \Phi_{6'7}\}$ living on intersections preserving $\mathcal{N} = 1$ supersymmetry. As we have already mentioned, for isolated brane intersections these massless fields indicate that the complex structure moduli are chosen such that one sits on a line of marginal stability. On one side of this line, the intersecting branes break supersymmetry without developing a tachyonic mode. This indicates that the intersecting brane configuration is stable. But on the other side of the line, the former massless chiral field becomes tachyonic and after condensation leads to a new in general non-flat supersymmetric brane wrapping a special Lagrangian 3-cycle. Since the tachyon transforms in the bifundamental representation, on this brane the gauge symmetry is broken to its diagonal subgroup.

We therefore expect for our compact situation that at least locally these bifundamental chiral multiplets indicate the existence of a recombined brane of the same volume but with the gauge group broken to the diagonal subgroup. In order to make our argument save, we need to show that the D-terms allow, that for certain continuous deformations of the complex structure moduli, just the fields $\{\Phi_{2'3}, \Phi_{2'4}, \Phi_{3'4}, \Phi_{5'6}, \Phi_{5'7}, \Phi_{6'7}\}$ become tachyonic, leaving the VEVs of the remaining $\{\Phi_{1'2}, \Phi_{1'3}, \Phi_{1'4}, \Phi_{1'5}, \Phi_{1'6}, \Phi_{1'7}\}$ vanishing. Then the fields $\{\Phi_{2'3}, \Phi_{2'4}, \Phi_{3'4}, \Phi_{5'6}, \Phi_{5'7}, \Phi_{6'7}\}$ condense to a new supersymmetric ground state and the gauge symmetries $U(2)^3$ are broken to the diagonal $U(2)$s. From general arguments for open string models with $\mathcal{N} = 1$ supersymmetry, it is known that the complex-structure moduli only appear in the D-term potential, whereas the Kähler moduli only appear in the F-term potential [215, 213, 216, 217].

field	n	$U(4)\times U(2)^3\times U(2)^3$
H_{12}	1	$(4;\bar{2},1,1;1,1,1)+c.c.$
H_{13}	1	$(4;1,\bar{2},1;1,1,1)+c.c.$
H_{14}	1	$(4;1,1,\bar{2};1,1,1)+c.c.$
H_{15}	1	$(\bar{4};1,1,1;2,1,1)+c.c.$
H_{16}	1	$(\bar{4};1,1,1;1,2,1)+c.c.$
H_{17}	1	$(\bar{4};1,1,1;1,1,2)+c.c.$
H_{25}	1	$(1;2,1,1;\bar{2},1,1)+c.c.$
H_{26}	1	$(1;2,1,1;1,\bar{2},1)+c.c.$
H_{27}	1	$(1;2,1,1;1,1,\bar{2})+c.c.$
H_{35}	1	$(1;1,2,1;\bar{2},1,1)+c.c.$
H_{36}	1	$(1;1,2,1;1,\bar{2},1)+c.c.$
H_{37}	1	$(1;1,2,1;1,1,\bar{2})+c.c.$
H_{45}	1	$(1;1,1,2;\bar{2},1,1)+c.c.$
H_{46}	1	$(1;1,1,2;1,\bar{2},1)+c.c.$
H_{47}	1	$(1;1,1,2;1,1,\bar{2})+c.c.$

Table 7.8: Non-chiral spectrum (Higgs fields)

Remember that the Green-Schwarz mechanism requires the Chern-Simons couplings to be of the form

$$S_{\mathrm{CS}} = \sum_{i=1}^{b_3}\sum_{a=1}^{k}\int d^4x\, M_{ai}\, B_i \wedge \frac{1}{N_a}\,\mathrm{tr}(F_a) \tag{7.46}$$

The supersymmetric completion involves a coupling of the auxiliary field D_a

$$S_{\mathrm{FI}} = \sum_{i=1}^{b_3}\sum_{a=1}^{k}\int d^4x\, M_{ai}\, \frac{\partial\mathcal{K}}{\partial\phi_i}\,\frac{1}{N_a}\,\mathrm{tr}(D_a) \tag{7.47}$$

where ϕ_i are the super-partners of the Hodge duals of the RR 2-forms and $\mathcal{K}$ denotes the Kähler potential. Thus, these couplings give rise to FI-terms depending on the complex structure moduli which we parameterize simply by $\mathcal{B}^i \equiv \partial\mathcal{K}/\partial\phi_i$ [8].

Let us now discuss the D-term potential for the $U(4)\times U(2)^3\times U(2)^3$ gauge fields in our model and see whether it allows supersymmetric ground states

[8]For our purposes we do not need the precise form of the Kähler potential as long as the map from the complex structure moduli ϕ_i to the new parameters $\mathcal{B}^i$ is one to one. But this is the case, at least for a sufficiently small open set $U \ni \mathcal{B}^i$, as the functional determinant for the map between these two sets of variables is equal to $\det\left(\frac{\partial^2\mathcal{K}}{\partial\phi_i\partial\phi_j}\right)$, which is non-vanishing for a positive definite metric on the complex structure moduli space (i and j including both holomorphic and antiholomorphic indices).

of the type described above. The D-term potential including only the chiral matter and the FI-terms in general reads

$$\begin{aligned} V_D &= \sum_{a=1}^{k} \sum_{r,s=1}^{N_a} \frac{1}{2g_a^2} (D_a^{rs})^2 \\ &= \sum_{a=1}^{k} \sum_{r,s=1}^{N_a} \frac{1}{2g_a^2} \left(\sum_{b=1}^{k} \sum_{p=1}^{N_b} q_{ab}\, \Phi_{ab}^{rp}\, \bar{\Phi}_{ab}^{sp} + g_a^2 \sum_{i=1}^{b_3} \frac{M_{ai}}{N_a}\, \mathcal{B}^i\, \delta^{rs} \right)^2 \end{aligned} \tag{7.48}$$

where the indices (r, s) numerate the N_a^2 gauge fields in the adjoint representation of the gauge factor $U(N_a)$ and the sum over b is over all chiral fields charged under $U(N_a)$. The gauge coupling constants g_a depend on the complex structure moduli as well, but since we are only interested in the leading order effects, we can set them to constant values on the line of marginal stability. Since all branes *in our particular model* have the same volume there, in the following we set them to the same constant:

$$g_a = g \qquad \forall a \in \{1, \ldots 7\} \tag{7.49}$$

To make things simpler we will set them to one.[9] For the time being, we are only interested in breaking the $U(2)^3 \times U(2)^3$ part of the gauge group down to the diagonal $U(2) \times U(2)$ by deforming the closed string background. We achieve this by giving appropriate VEVs to the fields $\mathcal{B}^i$ in the D-term potential (7.48).

In our case, the charges q_{ab} can be read off from table 7.7 [10] and the Green-Schwarz couplings M_{ai} from table 7.6 using the definition (7.39).

With the explicit D-term potential at hand, we will sketch the argument, why expectation values for the fields Φ_{ab} with a and b belonging to two different gauge groups, will break the gauge symmetry from $U(N)_a \times U(N)_b$ to a diagonal subgroup $U(N)_\alpha$. First, we will make a comment on the possible form of the $(\Phi_{ab})^{rp}$. We note that $\sum_{p=1}^{N_b} \Phi_{ab}^{rp} \bar{\Phi}_{ab}^{sp} = (\Phi_{ab}\Phi_{ab}^\dagger)^{rs}$ is a hermitian matrix. If only one such matrix is present, then δ^{rs} in (7.48) always forces $\Phi_{ab}\Phi_{ab}^\dagger$ to be a diagonal matrix. Depending on the exact form of the D-term, and especially in the example at hand, the hermitian matrices $\Phi_{ab}\Phi_{ab}^\dagger$ might be forced to diagonal form, even if several of such bifundamental fields are present. This means that the vectors $C_{ab}^{(i)}$, $i = 1 \ldots N_a$, that form Φ_{ab} as row- or column-vectors, are orthogonal and normalized to $||C_{ab}^{(i)}|| = \chi_{ab}\ \forall i$. Consequently Φ_{ab} is a $U(N_a)$ matrix up to a constant factor χ_{ab}. With a further unitary transformation, this time involving both gauge groups, we can achieve:[11]

$$\Phi_{ab} = \chi_{ab} \mathbb{1}_{N_a}, \qquad \chi_{ab} \in \mathbb{R}_+ \tag{7.50}$$

[9] This simplification will not affect the following conclusions.

[10] The charge is $q_a = 1$ if the representation of the $U(N_a)$ is N_a and $q_a = -1$ if it is $\bar{N}_a$.

[11] The diagonal form is obtained by $\Phi_{ab} \to U_a \Phi_{ab} U_b$ with $U_b = U_a^{-1}$. Such a U_a always exists for one matrix Φ_{ab}. Potential phases can be eliminated, because one can transform Φ_{ab} by independent $U(N)_a$ and $U(N)_b$ matrices on both indices. However, with too many fields Φ_{ab} involved in the D-term a simultaneous diagonalization of all VEVs Φ_{ab} might fail for generic cases. In the case at hand the small number of fields allows for simultaneous diagonalization.

Assuming for the moment that we have only one field Φ_{ab} acquiring a VEV, we will show that it gives masses to a special linear combination of the vector potentials A_a^μ and A_b^μ, while leaving another combination massless, thereby breaking the $U(N_a) \times U(N_b) \to U(N_\alpha)$ (with $N_a = N_b = N_\alpha$). What we will present is a generalization of the usual $U(1)$ gauge symmetry breaking by a suitable D-term (i.e. a D-term with the right sign) to the case where the chiral fields transform in the bifundamental representation of $N_1 \times \bar{N}_2$. We split Φ_{ab} into its (diagonal) VEV $\langle \Phi_{ab} \rangle = \chi_{ab} \mathbb{1}_{N_a}$ and a field $C_{ab}(x)$ that describes the fluctuations around the VEV:[12]

$$\Phi_{ab} = U_a(x)\big(\chi_{ab} + C_{ab}\big)U_b^{-1}(x) \tag{7.51}$$

As we are just interested in the (gauge-invariance breaking) mass term of the gauge field we need to consider only the covariant derivative of Φ_{ab}:

$$\nabla_\mu \Phi_{ab} = \partial_\mu \Phi_{ab} - iA_\mu^a \Phi_{ab} + i\Phi_{ab} A_\mu^b \tag{7.52}$$

Next we will make a finite gauge change in the gauge potentials:

$$\begin{aligned} A_\mu^a &\to A'^a_\mu \equiv U_a^{-1}(x) A_\mu^a U_a(x) + i(\partial_\mu U_a(x)) U_a^{-1}(x) \\ A_\mu^b &\to A'^b_\mu \equiv U_b^{-1}(x) A_\mu^b U_b(x) + i(\partial_\mu U_b(x)) U_b^{-1}(x) \end{aligned} \tag{7.53}$$

Expressed in terms of the gauge transformed fields (7.53), the covariant derivative (7.52) becomes:

$$\nabla_\mu \Phi_{ab} = U_a(x)\big(\nabla'_\mu C_{ab} + i\chi_{ab}(A'^b_\mu - A'^a_\mu)\big)U_b^{-1}(x) \tag{7.54}$$

The kinetic term $\mathrm{tr}\left((\nabla\Phi_{ab})^2\right)$ leads to a mass term:

$$\mathrm{tr}\left((\nabla\Phi_{ab})^2\right) = \mathrm{tr}(\nabla' C_{ab})^2 + \chi_{ab}^2 \,\mathrm{tr} \begin{pmatrix} A'_a \\ \hline A'_b \end{pmatrix}^\dagger \left(\begin{array}{c|c} \mathbb{1}_{N_a} & -\mathbb{1}_{N_a} \\ \hline -\mathbb{1}_{N_a} & \mathbb{1}_{N_a} \end{array}\right) \begin{pmatrix} A'_a \\ \hline A'_b \end{pmatrix} \tag{7.55}$$

The mass matrix appearing in (7.55) has rank N_a, leaving the diagonal combination $A'_a + A'_b$ massless, while giving $A'_a - A'_b$ a mass of order χ_{ab} [13]. Thereby the gauge symmetry gets broken: $U(N_a) \times U(N_b) \to U(N_\alpha)$, $(N_a = N_b = N_\alpha)$. The kinetic term of the gauge field is invariant under the gauge transformation.[14] The generalization to an additional field Φ_{ac} with VEV $\chi_{ac}\mathbb{1}_{N_a}$ is straightforward and leads again to a mass term:

$$\mathrm{tr} \begin{pmatrix} A'_a \\ \hline A'_b \\ \hline A'_c \end{pmatrix}^\dagger \left(\begin{array}{c|c|c} (\chi_{ab}^2 + \chi_{ac}^2)\mathbb{1}_{N_a} & -\chi_{ab}^2 \mathbb{1}_{N_a} & -\chi_{ac}^2 \mathbb{1}_{N_a} \\ \hline -\chi_{ab}^2 \mathbb{1}_{N_a} & \chi_{ab}^2 \mathbb{1}_{N_a} & 0 \\ \hline -\chi_{ac}^2 \mathbb{1}_{N_a} & 0 & \chi_{ac}^2 \mathbb{1}_{N_a} \end{array}\right) \begin{pmatrix} A'_a \\ \hline A'_b \\ \hline A'_c \end{pmatrix} \tag{7.56}$$

The linear combination $A'_a + A'_b + A'_c$ stays massless, while the two other (orthogonal) linear combinations have mass² of the order $\chi_{ab}^2 + \chi_{ac}^2 \pm \sqrt{\chi_{ab}^4 - \chi_{ab}^2\chi_{ac}^2\chi_{ac}^4}$.

---

[12] $U_{a,b}^{-1}(x)$ are x-dependent unitary matrices.

[13] If the gauge kinetic term is canonically normalized.

[14] Of course the gauge kinetic function $f(\phi)$ that multiplies the term $F \wedge *F$ has to be expressed in terms of (7.51), but it does *not* produce a mass term or even worse, a term compensating the mass term in (7.55).

It might be even possible to give a VEV $\chi_{bc}\mathbb{1}_{N_a}$ to a field Φ_{bc} transforming in the bifundamental $N_b \times \bar{N}_c$ of $U(N_b) \times U(N_c)$ (if it exists). The generalization of the mass matrix (7.56) is straightforward:

$$\left(\begin{array}{c|c|c} (\chi_{ab}^2+\chi_{ac}^2)\mathbb{1} & -\chi_{ab}^2\mathbb{1} & -\chi_{ac}^2\mathbb{1} \\ \hline -\chi_{ab}^2\mathbb{1} & \chi_{ab}^2\mathbb{1} & 0 \\ \hline -\chi_{ac}^2\mathbb{1} & 0 & \chi_{ac}^2\mathbb{1} \end{array}\right)$$
$$\longrightarrow \left(\begin{array}{c|c|c} (\chi_{ab}^2+\chi_{ac}^2)\mathbb{1} & -\chi_{ab}^2\mathbb{1} & -\chi_{ac}^2\mathbb{1} \\ \hline -\chi_{ab}^2\mathbb{1} & (\chi_{ab}^2+\chi_{bc}^2)\mathbb{1} & -\chi_{bc}^2\mathbb{1} \\ \hline -\chi_{ac}^2\mathbb{1} & -\chi_{bc}^2\mathbb{1} & (\chi_{ac}^2+\chi_{bc}^2)\mathbb{1} \end{array}\right) \quad (7.57)$$

leaving again $A'_a + A'_b + A'_c$ massless while giving masses to the other two orthogonal combinations.

From the fact that $(N_a)^{-1}M_{ai}$ (cf. (7.39), p. 178) can be written as

$$\frac{M_{ai}}{N_a} = (\mathbb{1} - \bar{\sigma})_i^{\ j} v_{a,j} \quad (7.58)$$

we deduce that the components of $\mathcal{B}^i$ that can contribute to the FI-term (7.48) belong to a vector space of dimension $b_3/2$.[15] It is the Eigenspace of $\left(\bar{\sigma}^T\right)_i^{\ j}$ with Eigenvalue -1. Actually in our seven stack example the rank of M_{ai} is only seven. We are now searching for vectors $\mathcal{B}$, that do not couple to the first gauge group which is the $U(4)$:

$$\mathcal{B}^i(\mathbb{1} - \bar{\sigma})_i^{\ j} v_{1,j} \stackrel{!}{=} 0 \quad (7.59)$$

This condition ensures the $U(4)$ to be unbroken as it forces vanishing VEVs for the fields $\Phi_{a\in\{1,1'\},b}$. (The $U(4)$ is an essential part of the Pati-Salam model.) Giving a VEV χ_{ab} to one of the remaining fields is equivalent to imposing that

$$V_c^{(ab)} = \frac{M_{ci}}{N_c}\mathcal{B}^i_{(ab)} \quad (7.60)$$

equals a certain vector $V^{(ab)}$. In this way we can associate to each VEV χ_{ab} a vector $\mathcal{B}^i_{(ab)}$. Linear combinations of $\sum_{ab} l_{ab}V^{ab}$, $l_{ab} > 0$, will then lead to VEVs $l_{ab}\chi_{ab}$ for the fields Φ_{ab}. In other words: the classical moduli space of the fields

$$\left\{\Phi_{2',3}, \Phi_{2',4}, \Phi_{3',4}, \Phi_{5',6}, \Phi_{5',7}, \Phi_{6',7}\right\} \quad (7.61)$$

is a cone of real dimension six.[16] This is only possible as the matrix M_{ai} is of rank seven, i.e. of maximal rank. As a consequence, the relation (7.60) gets invertible. Furthermore the chiral field-content of our model (c.f. table 7.6, p. 179) leads to an unambiguous mapping between VEVs of the six fields (7.61) and the vectors $V^{(ab)}$ in (7.60) (iff we impose the condition of unbroken $U(4)_1$, eq. (7.59)). Therefore we can give the fields in (7.61) arbitrary VEVs that are

[15]The rank of $\mathbb{1} - \bar{\sigma}$ is of this dimension.

[16]It is not a vector space, as negative coefficients l_{ab} multiplying the basis vectors would break supersymmetry as they hinder the D-term (7.48) to vanish.

proportional to the unit matrix $\mathbb{1}_{N_a}$. As explained above, giving non-vanishing VEVs to at least two of the fields $\left\{\Phi_{2',3}, \Phi_{2',4}, \Phi_{3',4}\right\}$ will lead to the diagonal gauge breaking:

$$U(2)_2 \times U(2)_3 \times U(2)_4 \to U(2)_b \tag{7.62}$$

Analogously giving VEVs to two or three of the fields $\left\{\Phi_{5',6}, \Phi_{5',7}, \Phi_{6',7}\right\}$ leads to the gauge breaking:

$$U(2)_5 \times U(2)_6 \times U(2)_7 \to U(2)_c \tag{7.63}$$

The string theoretic interpretation of this low energy description is as follows: There exists a supersymmetric configuration where the branes $\{\pi_2, \pi'_3, \pi'_4\}$ and similarly the branes $\{\pi_5, \pi'_6, \pi'_7\}$ have recombined into a single brane within the same homology class, thereby breaking the gauge symmetry from

$$U(4)_1 \times \big(U(2)_2 \times U(2)_3 \times U(2)_4\big) \times \big(U(2)_5 \times U(2)_6 \times U(2)_7\big) \tag{7.64}$$

down to

$$U(4)_a \times U(2)_b \times U(2)_c \tag{7.65}$$

which is a three stack model.

In what follows we will give non-trivial VEVS only to the fields

$$\langle\Phi_{2',3}\rangle > 0, \langle\Phi_{2',4}\rangle > 0 \quad \text{and} \quad \langle\Phi_{5',6}\rangle > 0, \langle\Phi_{5',7}\rangle > 0 \tag{7.66}$$

while leaving $\langle\Phi_{3',4}\rangle = \langle\Phi_{6',7}\rangle = 0$. This simplifies some of the following mass formulæ (cf. eq. (7.71) to (7.73)). We will make some comments about the changes that occur for non-vanishing VEVs $\langle\Phi_{3',4}\rangle > 0$ and $\langle\Phi_{6',7}\rangle > 0$.

7.5.3 Gauge symmetry breaking

After this recombination process we are left with only three stacks of D6-branes wrapping the homology cycles

$$\pi_a = \pi_1, \quad \pi_b = \pi_2 + \pi'_3 + \pi'_4, \quad \pi_c = \pi_5 + \pi'_6 + \pi'_7 \tag{7.67}$$

These branes are not factorizable but we have presented arguments ensuring that they preserve the same supersymmetry as the closed string sector and the former intersecting brane configuration.[17] The chiral spectrum for this now 3 stack model is shown in table 7.9. The intersection numbers $\pi'_{b,c} \circ \pi_{b,c}$ do not vanish any longer, therefore giving rise to chiral multiplets in the symmetric and anti-symmetric representation of the $U(2)$ gauge factors. Clearly, these chiral fields are needed in order to cancel the formal non-abelian $U(2)$ anomalies.

[17]Since we get chiral fields in the (anti-)symmetric representations after brane recombination, one might check if those intersection numbers can also be obtained by flat factorizable D-branes. Remember that we had the first assumption that there are no such chiral fields in the (anti-)symmetric representations. In fact, after an extensive computer search we have not been able to find a model with just factorizable D-branes generating the chiral spectrum of table 7.9.

field	n	$SU(4) \times SU(2) \times SU(2) \times U(1)^3$
Φ_{ab}	2	$(4, 2, 1)_{(1,-1,0)}$
$\Phi_{a'b}$	1	$(4, 2, 1)_{(1,1,0)}$
Φ_{ac}	2	$(\bar{4}, 1, 2)_{(-1,0,1)}$
$\Phi_{a'c}$	1	$(\bar{4}, 1, 2)_{(-1,0,-1)}$
$\Phi_{b'b}$	1	$(1, S + A, 1)_{(0,2,0)}$
$\Phi_{c'c}$	1	$(1, 1, \bar{S} + \bar{A})_{(0,0,-2)}$

Table 7.9: Chiral spectrum for 3 stack PS-model

Computing the mixed anomalies for this model, one finds that two $U(1)$ gauge factors are anomalous and that the only anomaly free combination is

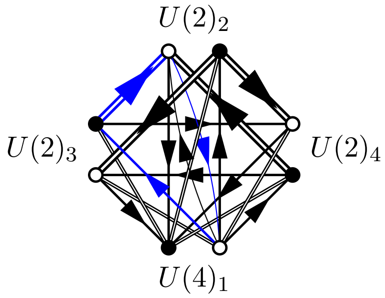
$$U(1) = U(1)_a - 3\,U(1)_b - 3\,U(1)_c \tag{7.68}$$

The quadratic axionic couplings reveal that the matrix M_{ai} in (7.39) has a trivial kernel and therefore all three $U(1)$ gauge groups become massive and survive as global symmetries. To summarize, after the recombination of some of the $U(2)$ branes we have found a supersymmetric 3 generation Pati-Salam model with gauge group $SU(4) \times SU(2)_L \times SU(2)_R$ which accommodates the Standard Model matter in addition to some exotic matter in the (anti-)symmetric representation of the $SU(2)$ gauge groups.

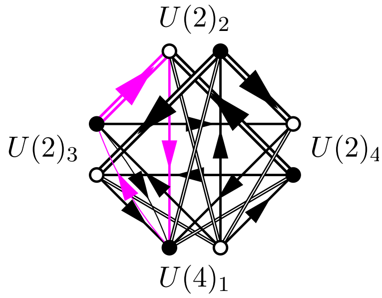
To compute the massless non-chiral spectrum after the recombination, we have to determine which Higgs fields receive a mass from couplings with the condensing chiral bifundamental fields. As we have explained earlier, the applicability of the low energy effective field theory is limited but still is the only information we have. So, we will see how far we can get. We first consider the sector of the branes $\{\pi_1, \ldots, \pi_4\}$ in figure 7.6. The chiral fields are indicated by an arrow and non-chiral fields by a double line without an arrow. The (chiral) fields which receive a VEV after small complex structure deformations are depicted by a double line with an arrow.

Let us decompose the Higgs fields inside one hyper-multiplet into its two chiral components $H_{1j} = \left(h_{1j}^{(1)}, h_{1j}^{(2)}\right)$ for $j = 2, 3, 4$. We observe a couple of closed triangles in the quiver diagram in figure 7.6 that give rise to the following Yukawa couplings in the super-potential:

$\phi_{2'3}\phi_{3\,1'}h_{1'2'}$:



$\phi_{2'3}\phi_{1\,2'}h_{3\,1}$:



(7.69)

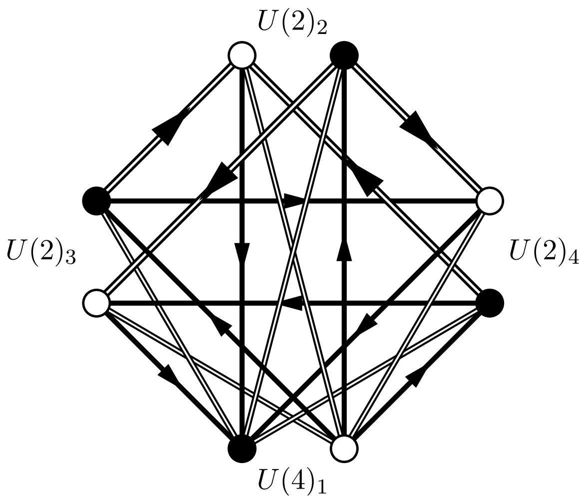
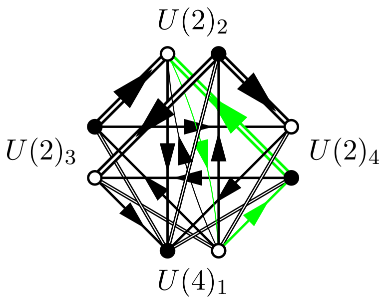


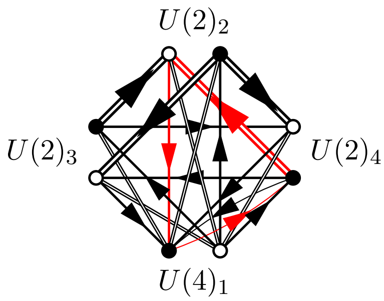
Figure 7.6: Quiver diagram for the branes $\{1,2,3,4\}$. The arrows denote a chiral multiplet. Double lines with arrows denote chiral multiplets where the scalar field gets a VEV due to a complex structure deformation (c.f. eq. (7.66)). Double lines without an arrow are hyper-multiplets. The filled dots describe a stack of branes and the unfilled dots are the $\Omega\bar\sigma$ images.

and

$\phi_{2'4}\phi_{41'}h_{1'2'}$:



$\phi_{2'4}\phi_{1\,2'}h_{41}$:



(7.70)

Condensation of the chiral fields $\Phi_{2'3}$ and $\Phi_{2'4}$ leads to a mass matrix for the six fields $\{\Phi_{1'2}, \Phi_{1'3}, \Phi_{1'4}, h^{(2)}_{12}, h^{(2)}_{13}, h^{(2)}_{14}\}$ of rank four. The mass terms look schematically:[18, 19]

$$\begin{pmatrix} h^{(2)}_{12} \\ h^{(2)}_{13} \\ h^{(2)}_{14} \\ \Phi_{1'2} \\ \Phi_{1'3} \\ \Phi_{1'4} \end{pmatrix}^{\dagger} \left(\begin{array}{ccc|ccc} & & & 0 & \langle\Phi_{2'3}\rangle & \langle\Phi_{2'4}\rangle \\ & 0 & & \langle\Phi_{2'3}\rangle & 0 & \big(\langle\Phi_{3'4}\rangle\big) \\ & & & \langle\Phi_{2'4}\rangle & \big(\langle\Phi_{3'4}\rangle\big) & 0 \\ \hline 0 & \langle\Phi_{2'3}\rangle & \langle\Phi_{2'4}\rangle & & & \\ \langle\Phi_{2'3}\rangle & 0 & \big(\langle\Phi_{3'4}\rangle\big) & & 0 & \\ \langle\Phi_{2'4}\rangle & \big(\langle\Phi_{3'4}\rangle\big) & 0 & & & \end{array}\right) \begin{pmatrix} h^{(2)}_{12} \\ h^{(2)}_{13} \\ h^{(2)}_{14} \\ \Phi_{1'2} \\ \Phi_{1'3} \\ \Phi_{1'4} \end{pmatrix} \qquad (7.71)$$

Thus, one combination of the three fields Φ, one combination of the three fields $h^{(2)}$ and furthermore the three fields $h^{(1)}$ remain massless. These modes just fit into the three chiral fields in table 7.9 in addition to one further hyper-multiplet in the $(4,2,1)$ representation of the Pati-Salam gauge group $U(4) \times U(2) \times$

[18] As strings with ends on $(a\,b)$ and $\bar\sigma$-pictures $(\bar\sigma(a)\bar\sigma(b)) = (a'b')$ (primes) are identified, we loosely identify the corresponding fields.

[19] We have included the possibility of a non-vanishing VEV $\langle\Phi_{3'4}\rangle > 0$ in parentheses. If all three VEVs are non-zero, the mass^2 matrix will be of rank six. In what follows, we assume $\langle\Phi_{3'4}\rangle = 0$.

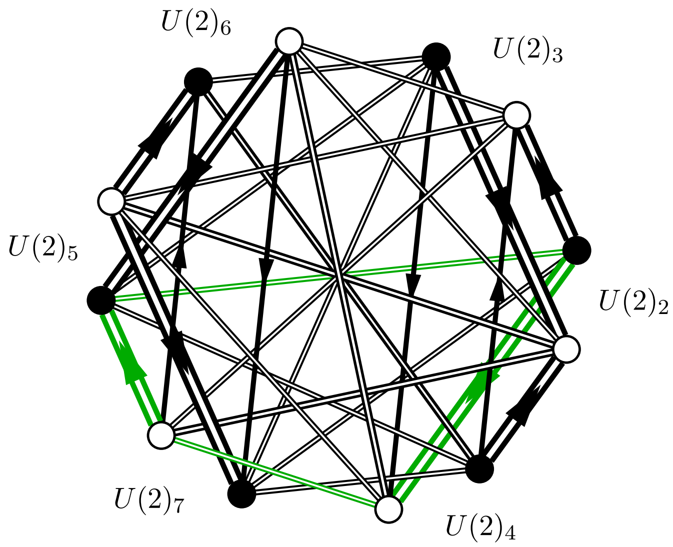


Figure 7.7: Quiver diagram for the branes $\{2, 3, 4, 5, 6, 7\}$. The right stacks $(U(2)_2 \ldots U(2)_4)$ and the left branes $(U(2)_2 \ldots U(2)_4)$ will combine into the right and left $U(2)$s of the left-right symmetric Pati-Salam Model after the scalar fields (7.66) get VEVs. The chiral multiplets that accommodate these condensed scalars are indicated by arrows (chirality) and double lines (VEV).

$U(2)$.[20] The condensation for the second triplet of $U(2)$s is completely analogous and leads to a massless hyper-multiplet in the $(4, 1, 2)$ representation.[21] The quiver diagram involving the six $U(2)$ gauge groups is shown in figure 7.7. In this quiver diagram the closed polygon $(2{-}4'{-}7'{-}5{-}2)$ (marked in green) generates a mass term after condensation of $\Phi_{2'3}$ and $\Phi_{5'7}$ for one chiral component inside $\{H_{25}, H_{47}\}$ and the sub-quiver $(2 - 4' - 5' - 7 - 2)$ (not marked) generates a mass term for one chiral component in $\{H_{46}, H_{47}\}$ (Remember that a hyper-multiplet consists of two chiral multiplets of opposite charge, $H = (h^{(1)}, h^{(2)})$). In analogy the mass terms for all nine Hyper-multiplets are obtained from fig. 7.7, too. They have the form:

$$\Psi^\dagger \left(\begin{array}{c|c|c} 0 & M_1 & M_2 \\ \hline M_1 & 0 & 0 \\ \hline M_2 & 0 & 0 \end{array} \right) \Psi \tag{7.72}$$

Here we have defined:

$$\begin{aligned} M_1 &= \begin{pmatrix} 0 & \Phi_{2'3}\Phi_{5'6} & \Phi_{2'4}\Phi_{5'6} \\ \Phi_{2'3}\Phi_{5'6} & 0 & 0 \\ \Phi_{2'4}\Phi_{5'6} & 0 & 0 \end{pmatrix} \\ M_2 &= \begin{pmatrix} 0 & \Phi_{2'3}\Phi_{5'7} & \Phi_{2'4}\Phi_{5'7} \\ \Phi_{2'3}\Phi_{5'7} & 0 & 0 \\ \Phi_{2'4}\Phi_{5'7} & 0 & 0 \end{pmatrix} \\ \Psi^T &= \left(\begin{array}{ccc|ccc|ccc} h_{25} & h_{35} & h_{45} & h_{26} & h_{36} & h_{46} & h_{27} & h_{37} & h_{47} \end{array} \right) \end{aligned} \tag{7.73}$$

[20] This hyper-multiplet would gain a mass if we give a VEV to the field $\Phi_{3'4}$.

[21] The hyper-multiplet in the $(4, 1, 2)$ representation would become massive if we give $\Phi_{6',7}$ a VEV.

field	n	$U(4)\times U(2)\times U(2)$
H_{aa}	1	$(\mathrm{Adj},1,1)+c.c.$
H_{bb}	1	$(1,\mathrm{Adj},1)+c.c.$
H_{cc}	1	$(1,1,\mathrm{Adj})+c.c.$
$H_{a'b}$	1	$(4,2,1)+c.c.$
$H_{a'c}$	1	$(4,1,2)+c.c.$
H_{bc}	3	$(1,2,\bar{2})+c.c.$

Table 7.10: Non-chiral spectrum for 3 stack PS-model

The mass matrix (7.72) for the chiral fields has rank six, so that three combinations of the four chiral fields, $h^{(1)}$, in $\{H_{36}, H_{37}, H_{46}, H_{47}\}$ remain massless. Since the intersection numbers in table 7.9 tell us that there are no chiral fields in the $(1,2,2)$ representation of the $U(4)\times U(2)\times U(2)$ gauge group, the other chiral components, $h^{(2)}$, of the hyper-multiplets must also gain a mass during brane recombination. A very similar behavior was found in [186], and it was pointed out that this might involve the condensation of massive string modes, as well. These would at least allow the correct mass terms in the quiver diagram. We expect that the quiver diagram really tells us half of the complete story, so that the non-chiral spectrum of the three generation Pati-Salam model is as listed in table 7.10. Intriguingly, these are just appropriate Higgs fields to break the Pati-Salam gauge group down to the Standard Model.

7.5.4 Getting the Standard Model

It is beyond the scope of this chapter to discuss all the phenomenological consequences of this 3 generation Pati-Salam model. However, we would like to present two possible ways of breaking the GUT Pati-Salam model down to the Standard Model.

7.5.4.1 Adjoint Pati-Salam breaking

There are still the adjoint scalars related to the unconstrained positions of the branes on the third T^2. By moving one of the four D6-branes away from the $U(4)$ stack, or in other words by giving VEVs to appropriate fields in the adjoint of $U(4)$, we can break the gauge group down to $U(3)\times U(2)\times U(2)\times U(1)$. Indeed the resulting spectrum as shown in table 7.11 looks like a three generation left-right symmetric extension of the Standard Model. Performing the anomaly analysis, one finds two anomaly free $U(1)$s, of which the combination $\frac{1}{3}(U(1)_1 - 3U(1)_4)$ remains massless even after the Green-Schwarz mechanism. This linear combination in fact is the $U(1)_{B-L}$ symmetry, which is expected to be anomaly-free in a model with right-handed neutrinos.

By giving a VEV to fields in the adjoint of $U(2)_R$, one obtains the next symmetry breaking, where the two $U(2)_R$ branes split into two $U(1)$ branes. This gives rise to the gauge symmetry $U(3)\times U(2)_L\times U(1)_R\times U(1)_R\times U(1)$.

n	$SU(3)_c \times SU(2)_L \times SU(2)_R \times U(1)^4$	$U(1)_{B-L}$
1	$(3,2,1)_{(1,1,0,0)}$	$\frac{1}{3}$
2	$(3,2,1)_{(1,-1,0,0)}$	$\frac{1}{3}$
1	$(\bar{3},1,2)_{(-1,0,-1,0)}$	$-\frac{1}{3}$
2	$(\bar{3},1,2)_{(-1,0,1,0)}$	$-\frac{1}{3}$
1	$(1,2,1)_{(0,1,0,1)}$	-1
2	$(1,2,1)_{(0,-1,0,1)}$	-1
1	$(1,1,2)_{(0,0,-1,-1)}$	1
2	$(1,1,2)_{(0,0,1,-1)}$	1
1	$(1,S+A,1)_{(0,2,0,0)}$	0
1	$(1,1,\bar{S}+\bar{A})_{(0,0,-2,0)}$	0

Table 7.11: Chiral spectrum for 4 stack left-right symmetric SM

In this case the following two $U(1)$ gauge factors remain massless after checking the Green-Schwarz couplings

$$\begin{aligned} U(1)_{B-L} &= \frac{1}{3}(U(1)_1 - 3U(1)_5) \\ U(1)_Y &= \frac{1}{3}U(1)_1 + U(1)_3 - U(1)_4 - U(1)_5 \end{aligned} \tag{7.74}$$

It is very assuring that we indeed obtain a massless hypercharge. The final supersymmetric chiral spectrum is listed in table 7.12 with respect to the unbroken gauge symmetries.

The anomalous $U(1)_1$ can be identified with the baryon number operator and survives the Green-Schwarz mechanism as a global symmetry. Therefore, in this model the baryon number is conserved and the proton is stable. Similarly, $U(1)_5$ can be identified with the lepton number and also survives as a global symmetry. To break the gauge symmetry $U(1)_{B-L}$, one can recombine the third and the fifth stack of D6 branes, which is expected to correspond to giving a VEV to the Higgs field $H_{3'5}$. We will see in section 7.5.4.2 that this brane recombination gives a mass to the right-handed neutrino.

To proceed, let us compute the relation between the Standard Model gauge couplings at the PS-breaking scale at string tree level. The $U(N_a)$ gauge couplings for D6-branes are given by

$$\frac{4\pi}{g_a^2} = \frac{M_s^3}{g_s}\mathrm{Vol}(\mathrm{D6}_a) \tag{7.75}$$

where $\mathrm{Vol}(\mathrm{D6}_a)$ denotes the internal volume of the 3-cycle the D6-branes are wrapping on. During the brane recombination process the volume of the recombined brane is equal to the sum of the volumes of the two intersecting branes. Therefore, we have the following ratios for the volumes of the five stacks of

n	field	$SU(3)\times SU(2)\times U(1)^3$	$U(1)_Y\times U(1)_{B-L}$
1	q_L	$(3,2)_{(1,1,0,0,0)}$	$(\frac{1}{3},\frac{1}{3})$
2	q_L	$(3,2)_{(1,-1,0,0,0)}$	$(\frac{1}{3},\frac{1}{3})$
1	u_R	$(\bar{3},1)_{(-1,0,-1,0,0)}$	$(-\frac{4}{3},-\frac{1}{3})$
2	u_R	$(\bar{3},1)_{(-1,0,0,1,0)}$	$(-\frac{4}{3},-\frac{1}{3})$
2	d_R	$(\bar{3},1)_{(-1,0,1,0,0)}$	$(\frac{2}{3},-\frac{1}{3})$
1	d_R	$(\bar{3},1)_{(-1,0,0,-1,0)}$	$(\frac{2}{3},-\frac{1}{3})$
1	l_L	$(1,2)_{(0,1,0,0,1)}$	$(-1,-1)$
2	l_L	$(1,2)_{(0,-1,0,0,1)}$	$(-1,-1)$
2	e_R	$(1,1)_{(0,0,1,0,-1)}$	$(2,1)$
1	e_R	$(1,1)_{(0,0,0,-1,-1)}$	$(2,1)$
1	ν_R	$(1,1)_{(0,0,-1,0,-1)}$	$(0,1)$
2	ν_R	$(1,1)_{(0,0,0,1,-1)}$	$(0,1)$
1		$(1,S+A)_{(0,2,0,0,0)}$	$(0,0)$
1		$(1,1)_{(0,0,-2,0,0)}$	$(-2,0)$
1		$(1,1)_{(0,0,0,-2,0)}$	$(2,0)$
2		$(1,1)_{(0,0,-1,-1,0)}$	$(0,0)$

Table 7.12: Chiral spectrum for 5 stack SM

D6-branes in our model

$$\mathrm{Vol}(\mathrm{D6}_2)=\mathrm{Vol}(\mathrm{D6}_3)=\mathrm{Vol}(\mathrm{D6}_4)=3\mathrm{Vol}(\mathrm{D6}_1),\quad \mathrm{Vol}(\mathrm{D6}_5)=\mathrm{Vol}(\mathrm{D6}_1) \tag{7.76}$$

This allows us at string tree level to determine the ratio of the Standard Model gauge couplings at the PS-breaking scale to be

$$\frac{\alpha_s}{\alpha_Y}=\frac{11}{3},\qquad \frac{\alpha_w}{\alpha_Y}=\frac{11}{9} \tag{7.77}$$

leading to a Weinberg angle $\sin^2\theta_w=9/20$ which differs from the usual $SU(5)$ GUT prediction $\sin^2(\theta_w)=3/8$. Encouragingly, from (7.77) we get the right order for the sizes of the Standard Model gauge couplings constants, $\alpha_s>\alpha_w>\alpha_Y$. It would be interesting to analyze whether this GUT value is consistent with the low energy data at the weak scale. A potential problem is the appearance of colored Higgs fields in table 7.10, which would spoil the asymptotic freedom of the $SU(3)$. In order to improve this situation one needs a model with less non-chiral matter, i.e. a model where not so many open string sectors actually preserve $\mathcal{N}=2$ supersymmetry.

7.5.4.2 Bifundamental Pati-Salam breaking

We can also use directly the bifundamental Higgs fields like $H_{a'c}$ to break the model down to the Standard Model gauge group. This higgsing in string theory

n	field	sector	$SU(3)_c \times SU(2)_L \times U(1)^4$	$U(1)_Y$
2	q_L	(AB)	$(3,2)_{(1,-1,0,0)}$	$\frac{1}{3}$
1	q_L	$(A'B)$	$(3,2)_{(1,1,0,0)}$	$\frac{1}{3}$
1	u_R	(AC)	$(\bar{3},1)_{(-1,0,1,0)}$	$-\frac{4}{3}$
2	d_R	$(A'C)$	$(\bar{3},1)_{(-1,0,-1,0)}$	$\frac{2}{3}$
2	u_R	(AD)	$(\bar{3},1)_{(-1,0,0,1)}$	$-\frac{4}{3}$
1	d_R	$(A'D)$	$(\bar{3},1)_{(-1,0,0,-1)}$	$\frac{2}{3}$
2	l_L	(BC)	$(1,2)_{(0,-1,1,0)}$	-1
1	l_L	$(B'C)$	$(1,2)_{(0,1,1,0)}$	-1
1	e_R	$(C'D)$	$(1,1)_{(0,0,-1,-1)}$	2
1	e_R	$(C'C)$	$(1,1)_{(0,0,-2,0)}$	2
1	e_R	$(D'D)$	$(1,1)_{(0,0,0,-2)}$	2
1	S	$(B'B)$	$(1,S+A)_{(0,2,0,0)}$	0

Table 7.13: Chiral spectrum for 4 stack SM

should correspond to a recombination of one of the four D6-branes wrapping π_a with one of the branes wrapping π'_c. Thus, we get the following four stacks of D6-branes

$$\pi_A = \pi_a, \quad \pi_B = \pi_b, \quad \pi_C = \pi_a + \pi'_c, \quad \pi_D = \pi_c \tag{7.78}$$

supporting the initial gauge group $U(3)\times U(2)\times U(1)^2$. The tadpole cancellation conditions are still satisfied. One gets the chiral spectrum by computing the homological intersection numbers as shown in table 7.13. By computing the mixed anomalies, one finds that there are two anomalous $U(1)$ gauge factors and two anomaly free ones

$$\begin{aligned} U(1)_Y &= \frac{1}{3}U(1)_A - U(1)_C - U(1)_D \\ U(1)_K &= U(1)_A - 9\,U(1)_B + 9\,U(1)_C - 9\,U(1)_D \end{aligned} \tag{7.79}$$

Remarkably, the axionic couplings just leave the hypercharge massless, so that we finally get the Standard Model gauge group $SU(3)_C \times SU(2)_L \times U(1)_Y$. In this model only the baryon number generator can be identified with $U(1)_1$, whereas the lepton number is broken. Therefore, the proton is stable and lepton number violating couplings as Majorana mass terms are possible. Note, that there are no massless right-handed neutrinos in this model. As we have mentioned already, this model is related to the model discussed in the last section by a further brane recombination process, affecting the mass of the right-handed neutrinos. This brane recombination can be considered as a stringy mechanism to generate GUT scale masses for the right-handed neutrinos [186]. The different ways of gauge symmetry breaking that have been discussed so far are depicted in figure 7.8.

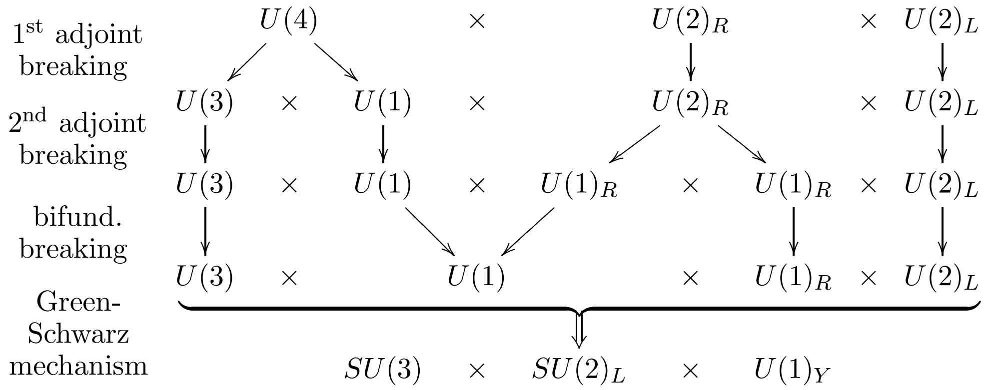


Figure 7.8: Gauge symmetry breaking of $U(4) \times U(2)_L \times U(2)_R$

It is evident from table 7.13 that there is also something unusually going on with the right-handed leptons. Only one of them is realized as a bifundamental field, the remaining two are given by symmetric representations of $U(1)$. This behavior surely will have consequences for the allowed couplings, in particular for the Yukawa couplings and the electroweak Higgs mechanism.

Computing the gauge couplings, we find the following ratios for the internal volumes of the four 3-cycles

$$\mathrm{Vol}(\mathrm{D6}_2) = \mathrm{Vol}(\mathrm{D6}_4) = 3\mathrm{Vol}(\mathrm{D6}_1), \quad \mathrm{Vol}(\mathrm{D6}_3) = 4\mathrm{Vol}(\mathrm{D6}_1) \tag{7.80}$$

This allows us to determine the ratio of the Standard Model gauge couplings at the GUT scale to be again

$$\frac{\alpha_s}{\alpha_Y} = \frac{11}{3}, \qquad \frac{\alpha_w}{\alpha_Y} = \frac{11}{9} \tag{7.81}$$

leading to a Weinberg angle $\sin^2\theta_w = 9/20$. Thus, both models provide the same prediction for the Weinberg-angle at the GUT scale.

7.5.4.3 Electroweak symmetry breaking

Finally, we would like to make some comments on electroweak symmetry breaking in this model. From the quiver diagram of the $U(4)\times U(2)\times U(2)$ Pati-Salam model we do not expect that the three Higgs fields in the $(1,\bar{2},2)$ representation get a mass during the brane recombination process. Therefore, our model does contain appropriate Higgs fields to participate in the electroweak symmetry breaking. The three Higgs fields, H_{bc}, in the Pati-Salam model in table 7.10 give rise to the Higgs fields

$$H_{BD} = (1,2)_{(0,1,0,-1)} + c.c.\,, \qquad H_{B'C} = (1,2)_{(0,1,1,0)} + c.c. \tag{7.82}$$

for the $SU(3)_c \times SU(2)_L \times U(1)_Y$ model above.

Of course supersymmetry should already be broken by some mechanism above the electroweak symmetry breaking scale, but nevertheless we can safely discuss the expectations from the purely topological data of the corresponding brane recombination process. Since we do not want to break the color $SU(3)$, we still take a stack of three D6-branes which are wrapped on the cycle $\pi_\alpha = \pi_A$. Giving a VEV to the fields H_{BD} is expected to correspond to the brane recombination

$$\pi_\beta = \pi_B + \pi_D \tag{7.83}$$

However, for the brane recombination

$$\pi_\gamma = \pi'_B + \pi_C \tag{7.84}$$

the identification with the corresponding field theory deformation is slightly more subtle, as the intersections between these two branes support both the massless chiral multiplet $l_L^{B'C}$ as listed in table 7.13 and the Higgs field $H_{B'C}$. Thus, the intersection preserves only $\mathcal{N} = 1$ supersymmetry and one might expect that some combination of $l_L^{B'C}$ and $H_{B'C}$ are involved in the brane recombination process. Even without knowing all the details, in the following we can safely compute the chiral spectrum via intersection numbers.

After the brane recombination we have a naive gauge group $U(3) \times U(1) \times U(1)$, which however is broken by the Green-Schwarz couplings to $SU(3)_c \times U(1)_{\text{em}}$ with

$$U(1)_{\text{em}} = \frac{1}{6}U(1)_\alpha - \frac{1}{2}U(1)_\beta + \frac{1}{2}U(1)_\gamma \tag{7.85}$$

Interestingly, just $U(1)_{\text{em}}$ survives this brane recombination process. Moreover, all intersection numbers vanish, so that there are no chiral massless fields, i.e. all quarks and leptons in table 7.13 have gained a mass including the left-handed neutrinos and the exotic matter. Looking at the charges in table 7.13, one realizes that in the leptonic sector this Higgs effect cannot be the usual one, where simply l_L and e_R receive a mass via some Yukawa couplings. Here also higher dimensional couplings, like the dimension five coupling

$$W \sim \frac{1}{M_s} \bar{H}_{BD}\, \bar{H}_{BD}\, S\, e_R^{D'D} \tag{7.86}$$

are relevant. These couplings induce a mixing of the Standard Model matter with the exotic field, S. Thus we can state, that by realizing some of the right-handed leptons in the (anti-)symmetric representation, the exotic field is needed to give all leptons a mass during electroweak symmetry breaking. It remains to be seen whether the induced masses can be consistent with the low-energy data.

Concluding remarks

In this chapter we have studied intersecting brane worlds for the $T^6/\mathbb{Z}_4$ orientifold background with special emphasis on supersymmetric configurations. We have found as a first non-trivial result a supersymmetric three generation Pati-Salam type extension of the Standard Model with some exotic matter. The chiral matter content is only slightly extended by one chiral multiplet in the (anti-)symmetric representation of $SU(2)_L$. The presence of this exotic matter can be traced back to the fact that we were starting with a Pati-Salam gauge group, where the anomaly constraints forced us to introduce additional matter. Issues which arose for non-supersymmetric models will also appear in the supersymmetric setting. Since the Green-Schwarz mechanism produces global $U(1)$ symmetries, the allowed couplings in the effective gauge theory are usually much more constrained than for the Standard Model.

With such model at hand, many phenomenological issues deserve to be studied, as for instance mechanisms for supersymmetry breaking, the generation of soft breaking terms, Yukawa and higher dimensional couplings, the generation of μ-terms and gauge coupling unification.[22] It also remains to be seen whether the electroweak Higgs effect indeed produces the correct masses for all quarks and leptons. Moreover, one should check whether the renormalization of the gauge couplings from the string respectively the PS-breaking scale down to the weak scale can lead to acceptable values for the Weinberg angle.[23]

The motivation for this analysis was to start a systematic search for realistic supersymmetric intersecting brane world models. We have worked out some of the technical model building aspects when one is dealing with more complicated orbifold backgrounds containing in particular twisted sector 3-cycles. These techniques can be directly generalized to, for instance, the $\mathbb{Z}_6$ orientifolds [57] or the $\mathbb{Z}_N \times \mathbb{Z}_M$ orientifold models [206].[24] It could be worthwhile to undertake a similar study for these orbifold models, too.

The final goal would be to find a realization of the MSSM in some simple intersecting brane world model. As should have become clear from our analysis, while phenomenologically interesting non-supersymmetric models are fairly easy to get, the same is not true for the supersymmetric ones. Requiring supersymmetry imposes very strong constraints on the possible configurations and as we have observed in the $\mathbb{Z}_4$ example, also the supply of possible intersection numbers is very limited. These obstructions appear to be less surprising, when one contemplates that for smooth backgrounds, by lifting to M-theory, the construction of an $\mathcal{N} = 1$ chiral intersecting brane world background with O6 planes and D6 branes is equivalent to the construction of a compact singular G_2 manifold. In this respect it would be interesting whether certain M-theory orbifold constructions like the one discussed in [222] are dual to the kind of models discussed in this chapter.

At a certain scale close to the TeV scale supersymmetry has to be broken. For the intersecting brane world scenario one might envision different mechanisms for such a breaking. First, we might use the conventional mechanism of gaugino condensation via some non-perturbative effect. Alternatively, one could

[22] Yukawa couplings for toroidal $\bar{\sigma}\Omega$-orientifolds have been calculated in [218] and [219] (also four-point couplings). However it is not clear if these calculations might be generalized to the case of recombined D-branes that appear in the models discussed in this chapter. The evolution of gauge-couplings including threshold corrections was investigated for supersymmetric $\bar{\sigma}\Omega$-orientifolds in [220].

[23] In [221] issues concerning gauge-coupling unification were addressed in the context of supersymmetric $\bar{\sigma}\Omega$-orientifolds with D6-branes.

[24] Meanwhile the $\mathbb{Z}_4 \times \mathbb{Z}_2$ $\bar{\sigma}\Omega$-orientifold with projection has been studied [168].

build models where the MSSM is localized on a number of D-branes, but where the RR-tadpole cancellation conditions requires the introduction of hidden sector branes, on which supersymmetry might be broken. This breaking could be mediated gravitationally to the Standard Model branes. A third possibility is to get D-term supersymmetry breaking by generating effective Fayet-Iliopoulos terms via complex structure deformations. We think that these issues and other phenomenological questions deserve to be studied in the future.

Conclusions

In this thesis we have investigated specific kinds of open-string theories. A striking feature of all constructions we considered is that they potentially contain chiral fermions. From the string theoretic point of view, chiral fermions arise due to non-trivial boundary conditions of the open-string. In the case of world-sheet supersymmetry the Ramond sector yields a reduced number of zero-modes. To be more specific, this could lead to a single zero-mode after GSO projection. By compactification to four space-time dimensions this results in a single Weyl fermion. However the number of world-sheet bosonic zero modes (e.g. intersection points or Landau levels) can be increased by the boundary conditions. As the Hilbert space of the string states is a product of world-sheet bosons and fermions: $\mathcal{H}_{\text{bos}} \otimes \mathcal{H}_{\text{ferm}}$, the degeneracy of world-sheet bosonic zero modes is inherited by the Ramond-sector. Furthermore one has to take into account, that there might appear space-time fermions of both chiralities. Therefore the multiplicity of world-sheet bosons encounters possible signs. As a result the total number of fermions with definite chirality is given by purely *topological* quantities like the intersection number of D-branes or the index of the twisted spin complex. In model-building this degeneracy is (roughly speaking) interpreted as the number of generations of a specific particle type. Therefore the degenerate states should be split by some mechanism[25] into states of different masses, but otherwise identical quantum numbers. By adjusting the topological data in a bottom-up approach, we could generate many phenomenologically appealing spectra.

As a basis for the subsequent chapters we quantized the open string with linear, but independent boundary conditions that are induced by D-branes of arbitrary dimension with constant NSNS B- and NS F-field(s) in chapter 4. We confirmed the disk result of Seiberg and Witten on the non-commutativity of open-string boundaries for the one-loop case (with arbitrary, constant $\mathcal{F}$-fluxes on the string-endpoints, but without Dirichlet conditions). Furthermore we investigated the zero- and momentum-mode spectrum in toroidal compactifications. It was shown that a kind of Landau degeneracy shows up, if the string end-points couple to different NS F-fields. This is an example of the degeneracy mentioned above.

Chapter 5 mainly reviews the content of our publication [1]. In this article we investigated the D-brane spectrum of asymmetric orbifolds and orientifolds.

[25] Such a mechanism should have a string-theoretic interpretation. For example different masses could arise due to Yukawa-couplings which are associated to some area of the world-sheet in intersecting brane-world models (cf. [218, 219]).

It turned out that left-right asymmetric twists imply in many cases the presence of D-branes with magnetic background fluxes, since the D-brane configuration has to be symmetrized under the (asymmetric) orbifold Group G. As an example we presented a space-time six-dimensional model that was obtained from orientifolding a $T^4/\mathbb{Z}_3^{\mathrm{L}} \times \mathbb{Z}_3^{\mathrm{R}}$-orbifold.[26] This article was finished before the work presented in chapter 4 was done. With some new insights gained in this chapter it was now possible to answer some so far open questions, like the quantization of the open-string momentum modes on toroidally compactified D-branes both for magnetized branes and lower dimensional branes from first principles.[27] This quantization was derived up to now only indirectly via the open-closed string correspondence (i.e. boundary states, cf. [223, 224]).

Chapter 6 is devoted to purely toroidal orientifolds. We considered both space-time six- and four-dimensional compactifications, i.e. compactifications on T^6 resp. T^4. Computations can be either done in the pure Ω-orientifold with D9-branes carrying NS $U(1)$-fluxes in the compact directions, or in the T-dual picture, where Ω is combined with complex conjugation $\bar{\sigma}$. Here the O-plane fills only a real subspace of the torus and its RR charge is canceled by D-branes of the same dimensionality. This means that we introduce D7-branes for the T^4-compactification and D6-branes on the six-torus T^6. Chiral fermions arise in the $\bar{\sigma}\Omega$-orientifold at the intersection points of the D7- resp. D6-branes. The number of chiral fermion generations is then given by the topological intersection numbers, while in the flux picture it is due to Landau degeneracies, which can be calculated by an index-theorem.[28] In order to get an interesting spectrum, one tries to distribute D-branes in such a way that a) they cancel the RR tadpole (thereby ensuring anomaly cancellation) and b) they yield the desired intersection numbers (resp. index). We were able to construct a four generation model with SM-like spectrum and gauge group $SU(3)\times SU(2)\times U(1)_Y \times U(1)^2$. The obstruction that the number of generations has to be even can be overcome by including so called **B**-type tori as shown in the subsequent publication [161].[29] However chiral configurations in purely toroidal constructions always break supersymmetry. Therefore the solutions might be unstable (i.e. divergent to a singular limit). Besides the Fischler-Susskind mechanism[30] there might exist further (yet unknown) mechanisms to

[26] We considered three variations of this orientifold. The main difference in these three different orientifolds is due to two alternative actions of the world sheet parity Ω on the zero- and momentum-modes. This is equivalent to different choices for the complex- and Kähler-structure of each of the two two-tori ($T^4 = T^2 \times T^2$). The two different choices result in three inequivalent orientifold models. One of these models was investigated before in [53] by means of conformal field theory. However a D-brane interpretation of the open-string sector could not be given in this former publication.

[27] The quantization for open strings on lower dimensional branes was done earlier for vanishing NSNS B-field.

[28] There are some subtleties, like the impossibility of obtaining the complete left-handed quark sector just from two kinds of branes (cf. section 6.4.2).

[29] **B**-type means on the in the "flux" picture that the NSNS B-field is set to $\alpha'/2$, which is still an Ω-symmetric background. In the T-dual "branes at angles" picture, the real part of the complex-structure τ of the **B**-type two-torus is fixed to $\tau_1 = 1/2$.

[30] It is however not clear, if the Fischler-Susskind mechanism does lead to a non-degenerate and non-supersymmetric limit.

stabilize the non-supersymmetric vacua.

Chapter 7 deals with the $\bar{\sigma}\Omega$-orientifold of an $\mathcal{N}=2$ supersymmetric $T^6/\mathbb{Z}_4$ orbifold. It is the second example besides the $\mathbb{Z}_2 \times \mathbb{Z}_2$ orientifold models (cf. [174]) that a $\bar{\sigma}\Omega$-orientifold admits chiral supersymmetric solutions. This chapter is mainly based on our publication [3], however more detailed in some points. We concentrate in the second half on an $U(4) \times U(2)^3_{\mathrm{L}} \times U(2)^3_{\mathrm{R}}$-model which we can break down to a three generation Pati Salam $U(4) \times U(2)_{\mathrm{L}} \times U(2)_{\mathrm{R}}$-model, while preserving supersymmetry. This is done by giving VEVs to fields in the low energy effective action. The chiral fermion spectrum in the latter model is given again by topological intersection numbers, while the non-chiral spectrum (i.e. hypermultiplets) is obtained by field theoretic considerations. Fields in the non-chiral spectrum serve as Higgs-particles for breaking the model down to an MSSM like spectrum.

While non-supersymmetric models have to deal with instabilities due to NSNS tadpoles (and potentially with open string tachyons, too), the supersymmetric models are stable.[31] However it is still an open issue how to break supersymmetry in a way that is manifestly compatible with string-theory.[32] The brane recombination process deserves a microscopic explanation.[33] Since special Lagrangian submanifolds play a prominent role in the construction of supersymmetric intersecting brane worlds, a richer knowledge about these objects is desirable, also in the more general context than Calabi-Yau orbifolds.

As another aim one could consider other toroidal orientifolds, searching again for a stringy realization of the MSSM.

We want to conclude with these few suggestions, even though many other questions related to this kind of orientifold constructions should be addressed, too.

[31] However instabilities might be induced by quantum or instanton corrections or so far unknown mechanisms.

[32] Field-theoretic considerations in related models do exist, some of them involving non-perturbative effects.

[33] This topic has recently been addressed by Hashimoto at the *Strings 2003* conference, however in a more general context.

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Hilfsmittel

Außer der angegebenen Literatur habe ich die Computeralgebra-Programme “Maple” und “Mathematica” verwendet. Ferner kam ein “C++” Compiler zum Einsatz. Die drei-dimensionalen Grafiken wurden mit dem “Ray-Tracing” Progamm “Povray” erstellt, wobei die Daten einiger drei-dimensionaler Objekte mit “Mathematica” berechnet wurden.

Selbständigkeitserklärung

Hiermit erkläre ich, die vorliegende Arbeit selbständig ohne fremde Hilfe verfaßt und nur die angegebene Literatur und Hilfsmittel verwendet zu haben.

Lars Görlich

11. August 2003

Appendix A

Theta-functions and related functions

A.1 η and ϑ-functions, identities and transformation under $SL(2,\mathbb{Z})$

The ϑ-functions are defined as follows

$$\vartheta \begin{bmatrix} a \\ b \end{bmatrix} (\tau) = \sum_{n\in\mathbb{Z}} q^{\frac{1}{2}(n+a)^2}\, \mathrm{e}^{2i\pi(n+a)b} \tag{A.1}$$

q is defined by $q = \mathrm{e}^{2\pi i\tau}$ ($\mathfrak{Im}\tau > 0$). The ϑ-functions admit the following representation as an infinite product:

$$\frac{\vartheta\left[{a \atop b}\right]}{\eta} = \mathrm{e}^{2i\pi ab}\, q^{\frac{1}{2}a^2-\frac{1}{24}} \prod_{n=1}^{\infty} \left(1 + q^{n+a-\frac{1}{2}}\mathrm{e}^{2i\pi b}\right)\left(1 + q^{n-a-\frac{1}{2}}\mathrm{e}^{-2i\pi b}\right), \tag{A.2}$$

with η being the Dedekind η-function:

$$\eta = q^{\frac{1}{24}} \prod_{n=1}^{\infty}(1-q^n) \tag{A.3}$$

A.1.1 Transformation under $SL(2,\mathbb{Z})$:

The ϑ- and η-functions transform under the generators S, T of the modular group $SL(2,\mathbb{Z})$ and under $P = TST^2S$ as follows:
S-transformation:

$$\tau \xrightarrow{S} -1/\tau \tag{A.4}$$

$$\frac{1}{\eta}\vartheta \begin{bmatrix} a \\ b \end{bmatrix} (\tau) = e^{2\pi i\, ab}\frac{1}{\eta}\vartheta \begin{bmatrix} b \\ -a \end{bmatrix} (-1/\tau) \qquad \eta(\tau) = \sqrt{\frac{i}{\tau}}\eta(-1/\tau) \tag{A.5}$$

T-transformation:

$$\tau \xrightarrow{T} \tau + 1 \tag{A.6}$$

$$\vartheta \begin{bmatrix} a \\ b \end{bmatrix} (\tau) = e^{i\pi(a^2 - a)} \vartheta \begin{bmatrix} a \\ b - a + 1/2 \end{bmatrix} (\tau + 1) \tag{A.7}$$

$$\eta(\tau) = e^{-i\pi/12} \eta(\tau + 1) \tag{A.8}$$

The following P-transformation prove useful in transforming Möbius-strip amplitudes from loop- to tree-channel. We have already inserted the Möbius-strip relation $t = \frac{1}{8l}$. The parameters a and b are not independent in the Möbius amplitude s.th. the phases get a simpler form.

P-Transformation:

$$P = T \circ S \circ T^2 \circ S \qquad \tau = it + \frac{1}{2} = \frac{i}{8l} + \frac{1}{2} \xrightarrow{P} \tau = i2l + \frac{1}{2} \qquad P \equiv T \circ S \circ T^2 \circ S \tag{A.9}$$

$$\frac{1}{\eta} \vartheta \begin{bmatrix} a \\ b \end{bmatrix} \left(\tau = it + \tfrac{1}{2}\right) = e^{-i\pi\left(\frac{3}{2} + a^2 + 2(2a+b)(b-1)\right)} \frac{1}{\eta} \vartheta \begin{bmatrix} 2b - a \\ b - a - 3/2 \end{bmatrix} \left(\tau = i2l + \tfrac{1}{2}\right) \tag{A.10}$$

$$\eta\left(\tau = it + \tfrac{1}{2}\right) = \sqrt{l}\, \eta\left(\tau = i2l + \tfrac{1}{2}\right) \tag{A.11}$$

A.1.2 Identities between ϑ-functions

The ϑ-functions obey several Riemannian identities [225]. Supersymmetry shows up in the partition functions by the vanishing of the vacuum-amplitudes. The phases of the different sectors $((NS, +)$, $(NS, -)$, $(R, +))$, which implicitly determine the GSO projection can be determined by these identities. For $u_1 + u_2 + u_3 = 0$ we have:

$$\sum_{\alpha,\beta \in \{0,1/2\}} \epsilon_{\alpha,\beta}\, \vartheta \begin{bmatrix} \alpha \\ \beta \end{bmatrix} \prod_{i=1}^{3} \vartheta \begin{bmatrix} \alpha \\ \beta + u_i \end{bmatrix} = 0 \tag{A.12}$$

$$\sum_{\alpha,\beta \in \{0,1/2\}} \epsilon_{\alpha,\beta}\, \vartheta \begin{bmatrix} \alpha \\ \beta \end{bmatrix} \vartheta \begin{bmatrix} \alpha \\ \beta + u_3 \end{bmatrix} \prod_{i=1}^{2} \vartheta \begin{bmatrix} \alpha + \frac{1}{2} \\ \beta + u_i \end{bmatrix} = 0 \tag{A.13}$$

$$\epsilon_{0,0} = 1, \quad \epsilon_{0,1/2} = \epsilon_{1/2,0} = -1$$

We set $u_3 = 0$ in the six dimensional models of chapter 6.

A.2 Poisson resummation formula for lattice sums

Sums of the following type are involved in traces over Kaluza Klein and winding contributions (for $\vec{a}, \vec{b} \in \mathbb{R}^d$, S a real symmetric and non degenerate $d \times d$

matrix):

$$\sum_{\vec{v}\in\mathbb{Z}^d} e^{i2\pi(\vec{v}+\vec{a})\cdot\vec{b}} e^{-\pi t\left((\vec{v}+\vec{a})^T\, S\, (\vec{v}+\vec{a})\right)}$$
$$= t^{-d/2}\frac{1}{\sqrt{\det S}} \sum_{\vec{w}\in\mathbb{Z}^d} e^{-i2\pi\vec{w}\cdot\vec{a}} e^{-\frac{\pi}{t}\left((\vec{w}+\vec{b})^T\, S^{-1}\, (\vec{w}+\vec{b})\right)} \tag{A.14}$$

A.3 Conformal blocks in $D = 6$

In this section we summarize the conformal blocks that we use to shorten the notation for the asymmetric $\mathbb{Z}_3^L \times \mathbb{Z}_3^R$ orbifold on $T^2 \times T^2$ in section 2.4 (p. 47) and its orientifolded version (or open descendants) in section 5.4 (p. 133). We define:[1]

$$\begin{aligned}
\rho_{00} &= \frac{1}{2}\sum_{\alpha,\beta=0,\frac{1}{2}} (-1)^{2\alpha+2\beta+4\alpha\beta} \frac{\vartheta\begin{bmatrix}\alpha\\ \beta\end{bmatrix}^4}{\eta^4}, \\
\rho_{0h} &= \frac{1}{2}\sum_{\alpha,\beta=0,\frac{1}{2}} (-1)^{2\alpha+2\beta+4\alpha\beta} \frac{\vartheta\begin{bmatrix}\alpha\\ \beta\end{bmatrix}^2}{\eta^2} \prod_{i=1}^{2} 2\sin(\pi h_i) \frac{\vartheta\begin{bmatrix}\alpha\\ \beta+h_i\end{bmatrix}}{\vartheta\begin{bmatrix}\frac{1}{2}\\ \frac{1}{2}+h_i\end{bmatrix}} && h \neq 0 \\
\rho_{gh} &= \frac{1}{2}\sum_{\alpha,\beta=0,\frac{1}{2}} (-1)^{2\alpha+2\beta+4\alpha\beta} \frac{\vartheta\begin{bmatrix}\alpha\\ \beta\end{bmatrix}^2}{\eta^2} \prod_{i=1}^{2} \frac{\vartheta\begin{bmatrix}\alpha+g_i\\ \beta+h_i\end{bmatrix}}{\vartheta\begin{bmatrix}\frac{1}{2}+g_i\\ \frac{1}{2}+h_i\end{bmatrix}} && g, h \neq 0
\end{aligned} \tag{A.15}$$

The functions (A.15) transform under $\tau \xrightarrow{S} -1/\tau$ like:

$$\begin{aligned}
\rho_{00} &\to & \rho_{00} & \\
\rho_{0h} &\to & (2\sin\pi h)^2\, \rho_{h0} & \qquad h \neq 0 \\
\rho_{h0} &\to & (2\sin\pi h)^{-2}\, \rho_{0,-h} & \qquad h \neq 0 \\
\rho_{gg} &\to & -\rho_{g,-g} & \qquad g \neq 0 \\
\rho_{g,-g} &\to & -\rho_{-g,-g} & \qquad g \neq 0
\end{aligned} \tag{A.16}$$

The modular T transformation $\tau \xrightarrow{T} \tau + 1$ acts by:

$$\frac{\rho_{gh}}{\eta^2} \xrightarrow{\tau\to\tau+1} \frac{\rho_{g,g+h}}{\eta^2} \tag{A.17}$$

As abbreviations for lattice sums we define (cf. [53]):

$$\begin{aligned}
\Lambda_{SU(3)^2} &\equiv \left(|\chi_0|^2 + |\chi_1| + |\chi_2|^2\right)^2 \\
\Lambda_R &\equiv \chi_0^2 \\
\Lambda_W^\omega &\equiv \left(\chi_0 + e^{i2\pi\omega/3}\chi_1 + e^{-i2\pi\omega/3}\chi_2\right)^2 \\
\Lambda_W &\equiv \Lambda_W^0
\end{aligned}$$

[1] We adopted our notation from [53] and restricted it to the case $d = 4$, i.e. compactification on a T^4.

where χ_0, χ_1 and χ_2 are the $SU(3)$ characters at level one defined by formula (2.113) (p. 48).

Appendix B

Equivalence classes of unitary symmetric and anti-symmetric matrices.

In this appendix we show that any symmetric or anti-symmetric unitary n-dimensional matrix U can be brought to the form (3.74) (or (3.75)) (page 76) via a transformation:

$$V^T U V \qquad V \in U(n) \tag{B.1}$$

Any unitary matrix U can be brought to diagonal form by conjugation:

$$\begin{gathered} \exists\, W,\ W \in U(n): \qquad \widetilde{U} = W^{-1} U W, \quad W \in U(n) \\ \widetilde{U} = \operatorname{diag}\bigl(e^{i\lambda_1} \dots e^{i\lambda_n}\bigr), \quad \lambda_i \in \mathbb{R} \end{gathered} \tag{B.2}$$

In any case we have for arbitrary vectors $e_i, f_j \in \mathbb{C}^n$ ($\langle \cdot, \cdot \rangle$ denotes the *hermitian* inner product):

$$\bigl\langle \overline{U} e_i, U f_j \bigr\rangle = \bigl\langle e_i, U^T U f_j \bigr\rangle \tag{B.3}$$

We also note that a basis change (B.1) corresponds to (note the complex conjugation in $\bar{d}_i$):

$$\begin{gathered} U_{ik} = \sum_{j,k=1}^{n} \bigl\langle \bar{d}_i, c_j \bigr\rangle \bigl\langle c_j, U c_k \bigr\rangle \bigl\langle c_k, d_l \bigr\rangle = \bigl\langle \bar{d}_i, U d_k \bigr\rangle \\ \bigl\langle c_i, c_j \bigr\rangle = \bigl\langle d_i, d_j \bigr\rangle = \delta_{ij} \end{gathered} \tag{B.4}$$

We choose the d_i to be Eigenvectors of U: $U d_i = \exp(i\lambda_i) d_i$. To incorporate both the symmetric and antisymmetric case we leave the phase arbitrary:

$$U = e^{i\phi} U^T \tag{B.5}$$

Inserting this result and $e_i = \bar{d}_i$, $f_j = d_j$ into (B.3) we get:

$$e^{i(\lambda_i - \lambda_j)} \bigl\langle \bar{d}_i, d_j \bigr\rangle = e^{i\phi} \bigl\langle \bar{d}_i, d_j \bigr\rangle \tag{B.6}$$

For $\phi \in \{0, \pi\}$ we see that $U_{ij} = \exp i\lambda_j \cdot \langle \bar{d}_i, d_j \rangle = 0$ iff $(\lambda_i - \lambda_j) \neq \phi \mod 2\pi$. This means that $\widetilde{U}$ is block-diagonal in this basis. Each block is associated with an Eigenvalue λ_j and the individual blocks $\Lambda_i(\lambda_i)$ are symmetric (resp. anti-symmetric) :

$$U = \begin{pmatrix} \Lambda_1(\lambda_1) & 0 & \cdots\cdots\cdots & & 0 \\ 0 & \boxed{\Lambda_2(\lambda_2)} & 0 & \cdots\cdots & 0 \\ \vdots & 0 & \ddots & & \vdots \\ \vdots & \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots\cdots\cdots & & \Lambda_s(\lambda_s) \end{pmatrix} \tag{B.7}$$

However the blocks Λ_i are in general neither proportional to the identity matrix nor to the standard symplectic form. However we can achieve this by finitely many repetitions of the described procedure:

a) As U is still unitary it can be diagonalized again by conjugation with a block-diagonal unitary matrix (cf. (B.2)).

b) We then transform U with the modified transformation (B.1) into this basis (B.4).

c) By induction we will reduce the size of the blocks Λ_i (Obviously the size can not grow since the unitary base transformations do not mix the different blocks Λ_i).

In the symmetric case $U = U^T$ this size reduction stops iff Λ_i is a matrix with Eigenvalue $\exp(i\lambda_j)$ for all of the vectors on which Λ_i acts non-trivially. This means that Λ_i is proportional to the identity matrix Id_{n_i}. By a transformation (B.1) with V acting block-diagonally on the separate blocks by $\exp(-i\lambda_i/2)\,\mathrm{Id}_{n_i}$, we transform Λ_i to a matrix with all vectors having Eigenvalue one. (That means: Λ_i is transformed to the identity). Applying this procedure to all blocks Λ_i gives the identity matrix $U = \mathrm{Id}_n$.

In the anti-symmetric case $U = -U^T$ this size reduction stops iff Λ_i is a block matrix acting non-trivially only on vectors with Eigenvalue λ_i and $-\lambda_i$. By a reordering of the basis (which is actually an $SO(n_i, \mathbb{Z})$ transformation.) we obtain:

$$\Lambda_i = e^{i\lambda_i} \begin{pmatrix} 0 & \mathrm{Id}_{n_i/2} \\ -\mathrm{Id}_{n_i/2} & 0 \end{pmatrix} \tag{B.8}$$

By a similar rescaling as in the symmetric case (i.e. by $V_i = \exp(-i\lambda_i/2)\,\mathrm{Id}_{n_i}$) U_i is seen to be of the standard symplectic form. Applying this procedure for all blocks Λ_i and reordering the basis, U can be transformed to standard symplectic from:

$$U = \begin{pmatrix} 0 & \mathrm{Id}_{n/2} \\ -\mathrm{Id}_{n/2} & 0 \end{pmatrix} \tag{B.9}$$

By a last transformation with $V = \exp(i\pi/2)$ we can transform U to the commonly used form (3.75) (p. 76).

We have proven that there exists only one equivalence class with respect to the transformation (B.1) for either a symmetric or a antisymmetric unitary matrix U. In the symmetric case this class can be represented by the identity matrix. In the anti-symmetric case this class can be represented by the standard symplectic form (B.9).

Appendix C

Spectrum and Eigenvectors of Lorentz transformations

It is rather well known that orthogonal matrices (which are a subset of the space of unitary matrices) can be diagonalized by unitary matrices. Their Eigenvectors make an orthogonal system, that might be normalized. The Eigenvalues have modulus one. We will now do the analogous classification for Lorentz transformations, i.e. those linear maps $\Lambda \in GL(n+1, \mathbb{R})$ that preserve the *minkowskian* metric G. By minkowskian we mean: one time and n space directions such that we have a light-cone. Surprisingly (or not) it turns out that the number of Eigenvectors might be lower than the dimension of space time. Our analysis is not valid for a metric with more time directions because this would ruin the structure of a (light)-cone which is essential in our proof.

Theorem 1 *A finite dimensional Lorentz transformation $\Lambda \in SO(1,n)$ preserving the corresponding metric G admits $n+1$ Eigenvectors, if there are no single light-like Eigenvectors with Eigenvalue ± 1. (This is a sufficient but, not a necessary condition). In this case $n-1$ of the Eigenvalues and Eigenvectors are complex (denoted by $\lambda_2 \dots \lambda_n$) with $|\lambda_i| = 1$ (including real as a subset) and two light-like and real with $\lambda_0 = \lambda_1^{-1}$ or one time-like Eigenvector with $\lambda_0 = \pm 1$ and n non-time-like Eigenvectors with Eigenvalue $|\lambda_i| = 1$.*

For the $SO(n)$ case the proof makes use of the fact that every Eigenvector has non-vanishing norm. Starting from one Eigenvector $v \in V$ (which always exists for non-degenerate maps in the *complexified* vector space $V^{\mathbb{C}}$) one can then build the orthogonal complement W of this Eigenvector. In our case we can build the orthogonal complement $V_v \equiv \{w \in V | \langle v, w \rangle = 0\}$ as long as $\|v\| \neq 0$:[1]

$$V_v = P_v(V) \tag{C.1}$$

with P_v being the projector defined by:

$$P_v(w) \equiv w - v\frac{\langle v, w \rangle}{\|v\|} \tag{C.2}$$

[1] As $v \in V^{\mathbb{C}}$ we also have to complexify the scalar product. By $\langle \,.\,,\,.\,\rangle$ we denote the hermitian version (i.e. complex conjugation on the first vector) of the real Minkowski scalar product.

We can now proceed by induction starting from one Eigenvector v:

a) If v is not light-like we note that λ_v has modulus one. As we have assumed Λ to be real (appropriate to our application in chapter 4) we also deduce that $\bar{v}$ is an Eigenvector with Eigenvalue $\lambda_{\bar{v}} = \bar{\lambda}_v$. We will then project onto $W = P_v(V)$ which is left invariant by Λ ($\langle v, \Lambda w\rangle = \lambda_v \langle \Lambda v, \Lambda w\rangle = \langle v, w\rangle = 0$).

b) If v is light-like its Eigenvalue is assumed to be real: λ complex would imply that $\bar{v}$ is also Eigenvector with Eigenvalue $\bar{\lambda}$. If $\langle v, \bar{v}\rangle \neq 0$ this implies $\lambda^2 = 1$. This implies that v is real (up to a phase). This contradicts $\langle v, \bar{v}\rangle \neq 0$. The second possibility, $\langle v, \bar{v}\rangle = 0$ implies that v and $\bar{v}$ are linear dependent and up to a phase: real. (One can see this by explicitly writing down the scalar product for two light-like vectors. If one normalizes the time component $v_0 = w_0 = a \in \mathbb{R}$ one discovers for the space component that the vectors should be perpendicular wrt. the minkowskian scalar product: $\langle \vec{v}, \vec{v}\rangle_{\text{herm.}} = \langle \vec{w}, \vec{w}\rangle_{\text{herm.}} = \langle \vec{w}, \vec{v}\rangle_{\text{herm.}}$. This implies that $\vec{v} = \vec{w}$. By $\langle\,.\,,\,.\,\rangle_{\text{herm.}}$ we mean the hermitian, positive definite scalar product of the space components.) We denote the Eigenvalue of the light-like v by λ_0. If $\lambda_0 \neq \pm 1$ we note that there is another light-like Eigenvector: As all non-light-like Eigenvectors only contribute Eigenvalues with $|\lambda| = 1$, in order for $\det \Lambda = \pm 1$ there has to exist at least one other light-like Eigenvector w with $\lambda_1 \equiv \lambda_w \neq \lambda_0$. Due to the structure of the light cone, w has non-vanishing scalar product with v. This implies $\lambda_1 = \lambda_0^{-1}\,(\in \mathbb{R})$. We can now linearly combine v and w into non-light-like vectors and project (by (C.2)) on the orthogonal complement which has two dimensions less. The orthogonal complement $W_{v,w}$ fulfills of course $\Lambda\big(W_{v,w}\big) \subset W_{v,w}$.

c) If v is light-like with Eigenvalue $\lambda = \pm 1$ the determinant argument does not apply. There might or might not be additional time-like Eigenvectors. If another linear indep. light-like Eigenvector exists, we could project out the space spanned by the two light-like Eigenvectors. If not, we can make no further general and simple statement, if other non-light-like Eigenvectors exist.[2] We simply have to exclude this latter case for our solution in chapter 4.

The procedure can be applied on the remaining subspace W until all Eigenvectors are found. There is one comment in order about the time-like Eigenvector which might exist. In principle it could have Eigenvalue $\lambda = \exp(i\phi), \phi \in \mathbb{R}$. However only $\lambda = \pm 1$ is actually possible: v time-like implies $\bar{v}$ time-like. Because there are no two orthogonal time-like directions, v is real up to a phase. This restricts the Eigenvalue of time-like Eigenvectors to be either plus or minus one.

[2] One could think to project by the projector $\mathcal{P}_\parallel$ or $\mathcal{P}_\perp$ (cf. eq. (4.13) ,p. 90) defined in chapter 4. The resulting two spaces have however non-vanishing scalar-products. The so defined complement of the light-like Eigenvector v: $W_v \equiv \ker(\mathbb{1} - \mathcal{P}_v)$ is not invariant under the Lorentz-transformation Λ: $\Lambda\big(W_v\big) \not\subset W_v$

Of course there can be maximally two lin. indep. light-like Eigenvectors v and w. This follows from considerations on the different scalar products and from the fact that linear independent light-like vectors have non-vanishing scalar products.

In the case that $\Lambda \in O(1, n+1)$ (and not only $SO(1, n+1)$) it can happen that a time-like Eigenvector with $\lambda = -1$ exists. This would correspond to a time reversal. Applied to the discussion on boundary conditions in chapter 4 this is interpreted as a brane localized in time, i.e. an instanton (of possibly higher space dimension).

We will now make some comments about the situation with only one light-like Eigenvector with Eigenvalue $\lambda = \pm 1$.
We consider the case $\lambda = +1$ first:
Investigations seem to exclude the degenerate case for $SO(1, n)$ rotations with $n < 4$. A light-like Eigenvector with $\lambda = +1$ implies (probably) always a second, lin. indep. light-like Eigenvector with $\lambda = +1$.[3] However in five space time dimensions one can construct Lorentz transformations by $\Lambda(F) = (G + F)^{-1}(G - F)$ with an antisymmetric F that has precisely one light-like Null-vector. The Null-vectors of F are the Eigenvectors of $\Lambda(F)$ with Eigenvalue one and vice versa. Numerical analysis gives some hints that in the case (i.e. $n = 4$) of precisely one light like Eigenvector the number of Eigenvectors decreases, but we can not yet make a definite statement in terms of a proof. As a quite general example we choose for $n = 4$:

$$F_{\mu\nu} = \begin{pmatrix} 0 & 0 & a & b & c \\ 0 & 0 & -a & -b & -c \\ -a & a & 0 & d & e \\ -b & b & -d & 0 & h \\ -c & c & -e & -h & 0 \end{pmatrix} \tag{C.3}$$

F has a one dimensional light-like Null-space $\lambda \cdot v$ with $v = (1, 1, 0, 0, 0)$. This leads to the following Jordan decomposition of the associated Lorentz transformation Λ:

$$\left(\begin{array}{c|cc} \begin{smallmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{smallmatrix} & \multicolumn{2}{c}{0} \\ \hline 0 & -\left(\frac{-1+d^2+e^2+h^2+2\sqrt{-d^2-e^2-h^2}}{1+d^2+e^2+h^2}\right) & 0 \\ & 0 & -\left(\frac{-1+d^2+e^2+h^2-2\sqrt{-d^2-e^2-h^2}}{1+d^2+e^2+h^2}\right) \end{array}\right) \tag{C.4}$$

Of course the base change in (C.4) destroys the property of preserving $G = (-1, 1, 1, 1, 1)$. However it shows that the number of Eigenvectors is smaller than

[3] We can not yet completely exclude this case as we made the Ansatz $\Lambda(F) = (G+F)^{-1}(G-F)$ with an antisymmetric F for a Lorentz rotation which is not the most general one. Also the Ansatz $\Lambda(\widetilde{F}) = \exp F$ (each $SO(1, n)^+$ transformation might be written that way [226], but not general $O(1, n)$ transformation) does not necessarily imply that each light-like Eigenvector of Λ is a Nullvector of $\widetilde{F}$. This one observes already from the fact that there exist $\widetilde{F} \neq 0$ which imply $\mathbb{1} = \exp \widetilde{F}$. In fact these are exactly those $\widetilde{F}$ with Eigenvalues $\lambda \in i2\pi\mathbb{Z}$ [227]. The statement we could make is that no antisymmetric matrices F exist below five dimensions which admit exactly one light-like Nullvector but no further Nullvector.

the dimension of space-time. It is also interesting that this effect only occurs above four space-time dimensions. We do not yet know if this degenerate case has physical consequences. At least our method for generating a general solution to the boundary conditions misses some dofs. It seems to be complicated to find the general solution and if possible to quantize it. We will leave this as a purpose for future work.

The case with exactly one light-like Eigenvector with Eigenvalue $\lambda = -1$ occurs if we consider a D-brane of the type described above at the $\sigma = 0$ end-point, and a light-like D-brane without any $\mathcal{F}$-field at the $\sigma = \pi$ end-point of the string, with the light-like Eigenvector of $(G+\mathcal{F})^{-1}(G-\mathcal{F})$ being perpendicular to the second brane. The R-matrix (cf. eq. (4.15), p. 90) would then reflect the light-like Eigenvector of the first matrix, resulting in exactly one light-like Eigenvector with Eigenvalue minus one. Therefore both the light-like $\lambda = 1$ and the light-like $\lambda = -1$ case are connected.

Appendix D

Quantities of the $\left(T^2 \times T^2 \times T^2\right)/\mathbb{Z}_4$-Orientifold

In this appendix we present some useful quantities of the $(T^2 \times T^2 \times T^2)/\mathbb{Z}_4$-orientifold that we discussed in chapter 7.

D.1 Orientifold planes

We present the results for the O6-planes and the action of $\Omega\bar{\sigma}$ on the homology lattice for the other three orientifold models. The result is summarized in table D.1. For the action of $\Omega\bar{\sigma}$ on the orbifold basis we find:

- **AAA**: For the toroidal 3-cycles we get

$$\begin{aligned} \rho_1 &\to \rho_1, \quad \bar{\rho}_1 \to -\bar{\rho}_1 \\ \rho_2 &\to -\rho_2, \quad \bar{\rho}_2 \to \bar{\rho}_2 \end{aligned} \tag{D.1}$$

  and for the exceptional cycles

$$\varepsilon_i \to \varepsilon_i \qquad \bar{\varepsilon}_i \to -\bar{\varepsilon}_i \qquad \forall i \in \{1, \dots, 6\} \tag{D.2}$$

- **AAB**: For the toroidal 3-cycles we get

$$\begin{aligned} \rho_1 &\to \rho_1, \quad \bar{\rho}_1 \to \rho_1 - \bar{\rho}_1 \\ \rho_2 &\to -\rho_2, \quad \bar{\rho}_2 \to -\rho_2 + \bar{\rho}_2 \end{aligned} \tag{D.3}$$

model	O6-plane
AAA	$4\,\rho_1 - 2\,\bar{\rho}_2$
AAB	$2\,\rho_1 + \rho_2 - 2\,\bar{\rho}_2$
ABA	$2\,\rho_1 + 2\,\rho_2 + 2\,\bar{\rho}_1 - 2\,\bar{\rho}_2$
ABB	$2\,\rho_2 + 2\,\bar{\rho}_1 - 2\,\bar{\rho}_2$

Table D.1: O6-planes of the $T^6/\mathbb{Z}_4$ orientifold and for the exceptional cycles

$$\varepsilon_i \to \varepsilon_i \qquad \bar{\varepsilon}_i \to \varepsilon_i - \bar{\varepsilon}_i \qquad \forall i \in \{1, \dots, 6\} \tag{D.4}$$

- **ABA**: For the toroidal 3-cycles we get

$$\begin{aligned} \rho_1 &\to \rho_2, \quad \bar{\rho}_1 \to -\bar{\rho}_2 \\ \rho_2 &\to \rho_1, \quad \bar{\rho}_2 \to -\bar{\rho}_1 \end{aligned} \tag{D.5}$$

and for the exceptional cycles

$$\begin{aligned} \varepsilon_1 &\to -\varepsilon_1 & \bar{\varepsilon}_1 &\to \bar{\varepsilon}_1 \\ \varepsilon_2 &\to -\varepsilon_2 & \bar{\varepsilon}_2 &\to \bar{\varepsilon}_2 \\ \varepsilon_3 &\to \varepsilon_3 & \bar{\varepsilon}_3 &\to -\bar{\varepsilon}_3 \\ \varepsilon_4 &\to \varepsilon_4 & \bar{\varepsilon}_4 &\to -\bar{\varepsilon}_4 \\ \varepsilon_5 &\to \varepsilon_6 & \bar{\varepsilon}_5 &\to -\bar{\varepsilon}_6 \\ \varepsilon_6 &\to \varepsilon_5 & \bar{\varepsilon}_6 &\to -\bar{\varepsilon}_5 \end{aligned} \tag{D.6}$$

D.2 Supersymmetry conditions

In this appendix we list the supersymmetry conditions for the remaining three orientifold models.

- **AAA**: The condition that such a D6-brane preserves the same supersymmetry as the orientifold plane is simply

$$\varphi_{a,1} + \varphi_{a,2} + \varphi_{a,3} = 0 \mod 2\pi \tag{D.7}$$

with

$$\tan\varphi_{a,1} = \frac{m_{a,1}}{n_{a,1}}, \quad \tan\varphi_{a,2} = \frac{m_{a,2}}{n_{a,2}}, \quad \tan\varphi_{a,3} = \frac{U_2\, m_{a,3}}{n_{a,3}} \tag{D.8}$$

This implies the following necessary condition in terms of the wrapping numbers

$$U_2 = -\frac{n_{a,3}}{m_{a,3}} \cdot \frac{(n_{a,1}\, m_{a,2} + m_{a,1}\, n_{a,2})}{(n_{a,1}\, n_{a,2} - m_{a,1}\, m_{a,2})} \tag{D.9}$$

- **AAB**: The condition that such a D6-brane preserves the same supersymmetry as the orientifold plane is simply

$$\varphi_{a,1} + \varphi_{a,2} + \varphi_{a,3} = 0 \mod 2\pi \tag{D.10}$$

with

$$\tan\varphi_{a,1} = \frac{m_{a,1}}{n_{a,1}}, \quad \tan\varphi_{a,2} = \frac{m_{a,2}}{n_{a,2}}, \quad \tan\varphi_{a,3} = \frac{U_2\, m_{a,3}}{n_{a,3} + \frac{1}{2} m_{a,3}} \tag{D.11}$$

This implies the following necessary condition in terms of the wrapping numbers

$$U_2 = -\frac{\left(n_{a,3} + \frac{1}{2} m_{a,3}\right)}{m_{a,3}} \cdot \frac{(n_{a,1}\, m_{a,2} + m_{a,1}\, n_{a,2})}{(n_{a,1}\, n_{a,2} - m_{a,1}\, m_{a,2})} \tag{D.12}$$

- **ABA**: The condition that such a D6-brane preserves the same supersymmetry as the orientifold plane is simply

$$\varphi_{a,1} + \varphi_{a,2} + \varphi_{a,3} = \frac{\pi}{4} \mod 2\pi \tag{D.13}$$

with

$$\tan\varphi_{a,1} = \frac{m_{a,1}}{n_{a,1}}, \quad \tan\varphi_{a,2} = \frac{m_{a,2}}{n_{a,2}}, \quad \tan\varphi_{a,3} = \frac{U_2\, m_{a,3}}{n_{a,3}} \tag{D.14}$$

This implies the following necessary condition in terms of the wrapping numbers

$$U_2 = \frac{n_{a,3}}{m_{a,3}} \cdot \frac{(n_{a,1}\, n_{a,2} - m_{a,1}\, m_{a,2} - n_{a,1}\, m_{a,2} - m_{a,1}\, n_{a,2})}{(n_{a,1}\, n_{a,2} - m_{a,1}\, m_{a,2} + n_{a,1}\, m_{a,2} + m_{a,1}\, n_{a,2})} \tag{D.15}$$

D.3 Fractional boundary states

The unnormalized boundary states in light cone gauge for D6-branes at angles in the untwisted sector are given by

$$\begin{aligned} |D;(n_I,m_I)\rangle_U =& |D;(n_I,m_I),\text{NSNS},\eta=1\rangle_U + |D;(n_I,m_I),\text{NSNS},\eta=-1\rangle_U \\ &+|D;(n_I,m_I),\text{RR},\eta=1\rangle_U + |D;(n_I,m_I),\text{RR},\eta=-1\rangle_U \end{aligned} \tag{D.16}$$

with the coherent state

$$\begin{aligned} &|D;(n_I,m_I),\eta\rangle \\ &= \int dk_2 dk_3 \sum_{\vec{r},\vec{s}} \exp\Big(-\sum_{\mu=2}^{3}\sum_{n>0} \frac{1}{n}\alpha^\mu_{-n}\tilde{\alpha}^\mu_{-n} \\ &\qquad - \sum_{I=1}^{3}\sum_{n>0}\frac{1}{2n}\Big(e^{2i\varphi_I}\zeta^I_{-n}\tilde{\zeta}^I_{-n} + e^{-2i\varphi_I}\bar{\zeta}^I_{-n}\tilde{\bar{\zeta}}^I_{-n}\Big) \\ &\qquad + i\eta\big[\text{fermions}\big]\Big)|\vec{r},\vec{s},\vec{k},\eta\rangle \end{aligned} \tag{D.17}$$

Here α^μ denotes the two real non-compact directions and ζ^I the three complex compact directions. The angles φ_I of the D6-brane relative to the horizontal axis on each of the three internal tori T^2 can be expressed by the wrapping numbers (n_I, m_I) as listed in appendix D.2. The boundary state (D.17) involves a sum over the internal Kaluza-Klein and winding ground states parameterized by $(\vec{r},\vec{s})$. The mass of these KK and winding modes on each T^2 in general reads

$$M_I^2 = \frac{|r_I + s_I\, U_I|^2}{U_{I,2}}\,\frac{|n_I + m_I\, T_I|^2}{T_{I,2}} \tag{D.18}$$

with $r_I, s_I \in \mathbb{Z}$ as above and U_I and T_I denote the complex and Kähler structure on the torus [2]. If the brane carries some discrete Wilson lines, $\vartheta = 1/2$, appropriate factors of the form $e^{isR\vartheta}$ have to be introduced into the winding sum in (5.21).

In the Θ^2 twisted sector, the boundary state involves the analogous sum over the fermionic spin structures (5.19) with

$$
\begin{aligned}
&|D;(n_I,m_I),e_{ij},\eta\rangle_T \\
&= \int dk_2 dk_3 \sum_{r_3,s_3} \exp\Big(-\sum_{\mu=2}^{3}\sum_{n>0}\frac{1}{n}\alpha^{\mu}_{-n}\tilde{\alpha}^{\mu}_{-n} \\
&\qquad\qquad - \sum_{I=1}^{2}\sum_{r\in\mathbb{Z}_0^+ + \frac{1}{2}} \frac{1}{2r}\left(e^{2i\varphi_I}\zeta^I_{-r}\tilde{\bar{\zeta}}^I_{-r} + e^{-2i\varphi_I}\bar{\zeta}^I_{-r}\tilde{\zeta}^I_{-r}\right) \\
&\qquad\qquad - \sum_{n>0}\frac{1}{2n}\left(e^{2i\varphi_3}\zeta^3_{-n}\tilde{\bar{\zeta}}^3_{-n} + e^{-2i\varphi_3}\bar{\zeta}^3_{-n}\tilde{\zeta}^3_{-n}\right) \\
&\qquad\qquad\qquad + i\eta\big[\text{fermions}\big]\Big)|r_3,s_3,\vec{k},e_{ij},\eta\rangle
\end{aligned}
\tag{D.19}
$$

where e_{ij} denote the 16 $\mathbb{Z}_2$ fixed points. Here, we have taken into account that the twisted boundary state can only have KK and winding modes on the third T^2 torus and that the bosonic modes on the two other T^2 tori carry half-integer modes.

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